Mathematical Tables and other Aids to Computation

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RAYMOND CLARE ARCHIBALD
DERRICK HENRY LEHMER

WITH THE COÖPERATION OF

LESLIE JOHN COMRIE
SOLOMON ACHILLOVICH JOFFE

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The Application of Commercial Calculating Machines to Scientific Computing

I propose to deal with one era in computing, but will first recall some of the landmarks in that art. Perhaps the first was the introduction of the decimal system. Then came the invention or discovery of logarithms in 1614 by NAPIER, which enabled multiplications and divisions to be done much more rapidly than had been the case before. Briggs' industry in tabulating logarithms of numbers and of trigonometrical functions made Napier's discovery immediately available to all computers.

Although PASCAL constructed the first calculating machine in 1642, and although LEIBNIZ, MORLAND, STANHOPE and others contributed ideas and designs, nearly two centuries were to elapse before a commercial calculating machine could be purchased. This was the arithmometer of THOMAS DE COLMAR, first used in actuarial offices in Paris in the 1820's. It uses the stopped-wheel principle enunciated by Leibnitz to determine which digit shall be added in each columnar position. There is little to report till about half a century later, when ODHNER invented his pin-wheel for conveying varying digits to the machine. This principle is still used in the Swedish Odhner and other barrel-type crank-driven hand machines, particularly the German Brunsviga, which acquired manufacturing rights about 1890. This marked the beginning of a period of development, which continued with ever-growing intensity till, for the second time this century, progress was interrupted by war. These developments include keyboard setting, electric drive, automatic control of multiplication and division, duplication of registers, and storage and transfer facilities.

Although these machines were developed primarily for the wide fields of commercial and accounting application, they lend themselves to mathematical and scientific computing, which consists, after all, of the four fundamental processes of addition, subtraction, multiplication and division —just the processes that any calculating machine is designed to perform. Between the two world wars comes the era under review—that in which scientists discovered more and more how numerical processes could be lightened by mass-produced commercial machines. To make such a discovery, it is necessary that one person should understand both the machine and the problem. As far as possible commercial machines should be used in their standard form, although minor alterations are permissible. The computer's art lies mainly in his ability to manipulate the problem and to apply ingenuity and low cunning in developing techniques that take advantage of the mechanical features of the machine.

If I express the view that this process of adaptation has gone further in Great Britain than in U. S. A., I should like to give a rational explanation of why this should be so. Our great advantage is that we have more material with which to operate. In U. S. A. there are only three outstanding crank-driven calculating machines—the Marchant, Monroe and Friden. These are similar in capacity and in possessing keyboard setting, electric drive, automatic multiplication and division, but few special features such as one finds in the German Brunsviga, Mercedes, Archimedes and Rheinmetall, or the

Swiss Madas, or the Swedish Facit. Perhaps a contributing factor has been the European willingness to use a large proportion of hand machines, for hand machines often have facilities and flexibility not possessed to the same degree by their more affluent electric cousins. I shall revert later to this theme in more detail, and mention some special features of machines and some special techniques.

The adaptation age has overlapped the beginning of another—namely that of the construction of special machines. I have sometimes felt that physicists and engineers are too prone to ask themselves "What physical, mechanical or electrical analogue can I find to the equation I have to solve?" and rush to the drawing board and lathe before enquiring whether any of the many machines that can be purchased over the counter will not do the job. Too often the only tangible result has been a pile of blue prints, and perhaps a single machine that reaches old age with teething troubles—a machine that benefits only a few people and often impoverishes the inventor, who pays for patents that he cannot sell, for modern manufacturers think in terms of thousands of machines. I do not for a moment contend that there should not be any special design or development, but I do urge that this should be preceded by an exhaustive attempt to utilise what exists, and that new development is proper only when what we now have leaves much to be desired, or when quantity or time considerations, or the relative cost of labour and machinery, point a definite finger. What would not be right in India, where subsistence requirements may almost be measured in handfuls of rice, can easily be justified if computers cost $2000 to $3000 a year plus supervision, administration and costly accommodation. In other words *look* at what has been provided by and for industry *before you leap* to the adventure of special design.

Outstanding among special developments is one for which there can be nothing but praise, namely differential analysers. Although differential equations can be (and are) solved by finite difference methods on existing machines, the quantity of low-accuracy solutions required today is such that time and cost would be prohibitive. The use of machines for handling infinitesimals rather than finite quantities has fully justified itself, and we may all feel grateful to VANNEVAR BUSH, whose pioneer work has been an inspiration to many followers.

We are now living in an age of development of relay and electronic computers designed to perform calculations on a scale and with a speed undreamt of a decade ago. Many of these still have to face their real life tests. They have been brought into being under the stress of military expediency and urgency. Some have arrived (or are arriving) too late for the war that is now happily concluded, and it remains to be seen whether future research can absorb their enormous output capacity at an economic level. It is only fair that the need for new machines should be judged, not in the light of what we now do, but in terms of what we could profitably do if they were available. In many cases they will enable numerical mathematical analysis, with its wide range of choice in parameters, to replace experiments with costly models.

To revert to our title. The prejudice against hand machines, or should we say the disposition to use electric machines, which is such a strong feature of American usage, is far less pronounced in Europe. Whilst this arises

partly from economic considerations, a contributing factor is the wider range of possibilities and flexibility provided. This is well illustrated by the Brunsviga, a pre-war favourite among scientists, especially for private exploratory work. Its outstanding feature (1925) is a device for transferring the contents of the product register to the setting levers, which correspond to the keyboard in electric machines. This lends itself directly to the formation of *abc* or *abcde* . . . without intermediate settings—a convenience that often outweighs the advantage of keyboard setting or electric drive. The application to $(a \pm b \pm c \cdots)x$ or $(ab \pm cd \pm ef \cdots)x$ is obvious. The operation $\frac{a}{bc}$ is performed by forming *bc*, transferring, and then using build-up division—exclusively a hand or non-automatic machine process—to produce *a* in the product register and the desired quotient in the multiplier register. Again, a negative result such as $\cdots$999 384 192 in the product register may be transferred; a backward turn then produces 615 808, which may then, if desired, be transferred to the setting levers to act as a multiplicand. It is difficult to exaggerate the usefulness of this feature, which I commend to the attention of future designers. It is sometimes combined with another feature, namely the ability to clear half the product register without disturbing the other half. Products formed in the right-hand portion of the product register may be transferred to the setting levers and, after a carriage movement, added or subtracted in the (temporarily) non-clearing left-hand or storage portion. This permits the formation of Σab (regardless of signs) with the examination of each individual product; the same is true of Σabc. Again, it permits summation from second differences without any intermediate writing or setting. Another application is the formation in one operation of moments Σfx and Σfx^2 from equidistant values of x.

The multiplier register of this machine has a unique device that enables it to show white figures for positive multiplications and red figures (with full carry over or tens transmission) for negative multiplications or for quotients in tear-down division. The great virtue of this device is that it has no lever, but is operated by the direction of the first turn after a clearance of the multiplier register—a backward turn slides a window and exposes the red figures.

Model 20 (circa 1933) of this machine has the large capacity $12\times11\times20$. The great advantage of 12 setting levers is that the machine lends itself to doing two small multiplications simultaneously, since there is room to separate the two multiplicands. An earlier variant (Model III) had two multiplier registers, one for individual multipliers or quotients, and one for their sum. Still another, called the Brunsviga Dupla (1928), with capacity $15\times10\times15$, had two product registers, one of which could be disconnected at will by a lever, and had white and red figures as in the multiplier register. The contents of either register could be transferred to the setting levers. This machine was used for summation from second differences, or for differencing a function directly to its second difference with a single setting of each function value. From a series of positive values of x and y, it could form Σx, Σy, Σx^2, Σxy, Σy^2 and check them in one run.

About 1930 there appeared a twin Brunsviga, made by connecting two $10\times8\times13$ machines so that they are driven by a common handle. The

left-hand machine can be turned in the same direction as the right-hand machine, or in the opposite direction, or it can be disengaged altogether from the drive. The obvious application is to the simultaneous multiplication of two multiplicands by the same multiplier; for example by setting $\sin\theta$ and $\cos\theta$ and multiplying by r, we get $x = r\sin\theta$ and $y = r\cos\theta$ together, one on each machine. With small 3-figure numbers four products can be formed together, i.e. the twins become quads. An expression of the form $\frac{ab}{c}$ can be evaluated in *one* operation; while the right-hand machine is forming $\frac{a}{c}$, the left-hand machine, which is rotating in the opposite direction, and on which b is set, is forming the product $b \times \frac{a}{c}$.

The original application of twin machines was to military rectangular co-ordinate survey; during the war intensive use was made of them for this purpose by both sides. Since the position of each point is defined by two co-ordinates, it is easily realised that techniques for determining x and y simultaneously can be developed. As an excellent illustration, suppose we know the co-ordinates of two points A and B and the bearings α and β to an unknown point P, whose co-ordinates are required; this is the intersection problem—the most fundamental in surveying. It is easily shown that

$$\begin{aligned} x_P &= x_A - y_A \tan\alpha + y_P \tan\alpha \\ &= x_B - y_B \tan\beta + y_P \tan\beta \end{aligned}$$

If a value of y_P can be found such that the two expressions on the right-hand side are equal to each other, they must both be equal to x_P. Hence one machine is devoted to the upper equation and one to the lower. After entering x_A and x_B in the two product registers, $\tan\alpha$ is set on its machine and multiplied by $-y_A$. This machine is then disengaged temporarily while $-y_B \tan\beta$ is added to the other product register. The two machines, which now have different quantities in their product registers, are once more connected, and the handle turned until they are balanced. Both product registers then show x_P and the multiplier register shows y_P.

Twin machines have also been applied successfully to the interpolation of pairs of double-entry tables, such as those giving rectangular co-ordinates with geographical co-ordinates (latitude and longitude) as argument. The inversion of such pairs, in order to interchange argument and respondent, is another achievement. All this points to their future use for making and interpolating tables of functions of complex variables—a problem in which numerical inversion may often be easier than analytical inversion.

During the war, when Brunsvigas could not be obtained, an attempt was made to twin the one and only British-made machine of this type—the Britannic, a copy of the Brunsviga produced 30 years ago during the first world war. The machine itself proved too unreliable for military conditions, so the problem was solved by twinning (in England) lease-lend hand Marchants, which proved entirely satisfactory. Thus it has come about that there are hundreds of twin Marchants in England, but none in U. S. A., where the single machines were born!

In the Facit machine lever setting is replaced by setting from a 10-key keyboard. This leads to an attractive, compact and inexpensive machine of

capacity $10 \times 10 \times 19$, at less than half the cost of an electric machine. The advantages of electric drive appear to me to be over-estimated; for many purposes I would prefer two hand machines to a single electric machine, especially when we remember that, with short-cutting in multiplication, a hand machine takes only 60 per cent of the number of revolutions taken by an automatic machine. The real advantage of an electric machine lies not so much in the lessening of muscular effort as in the facilities (if provided) for storage, transfer, accumulation, and automatic division.

The Mercedes Model 38 MSL, of capacity $8 \times 8 \times 16$, is a good example of a highly-developed electric calculating machine. Multiplication and division are automatic; products and multipliers can both be accumulated with full and independent control over signs. Any sum or product can become the multiplier of the next operation. There is a visible storage register, which can receive from or transfer to the product register. The two last-named facilities lend themselves immediately to the formation of Σabc.

The electric Madas is the third machine capable of doing Σabc without intermediate settings. Its peculiar form of automatic multiplication lends itself to forming Σa and Σa^2 with a single setting of each value of a. If two small numbers a and b are set, the machine will produce Σa, Σb, Σa^2, $2\Sigma ab$ and Σb^2 in one run—a delightful feature that should be appreciated by those who dabble in correlations. This facility also appeared later on the Monroe Model AA–1–213, which has also followed the German Hamann Selecta in incorporating automatic short-cutting. A useful and unique feature of this Monroe is the ability to call a stored constant multiplier at will instead of setting it when required; this is applicable to $x + ay$ where x and y are variable and a is constant.

Although trying to avoid favouritism, I must confess my admiration for the principle of the automatic multiplication of the American Marchant. In all other automatic machines, both multiplicand and multiplier must be set before multiplication actively begins; the fastest machines run at about 600–700 revolutions a minute. In the Marchant the multiplicand is set on the keyboard as usual, but the multiplier is entered by an auxiliary row of ten keys. As each key is pressed, its digit is absorbed into the multiplier and the carriage moved to the next position. While the machine is running on one digit, the next may be set. As the speed is equivalent to 1300 revolutions a minute, the product is always completed within half a second of the setting of the last digit of the multiplier. The high speed mentioned is possible because the numeral wheels are driven by continuous gears (selected by the keyboard setting) rather than engaged and disengaged for a fraction of a revolution according to the digit set.

The American Friden has followed the German Rheinmetall (sold in England as the Muldivo and in U. S. A. as the Mathematon) in adopting a 10-key set-up for the multiplier in automatic multiplication. However these machines do not have complete tens transmission in their product registers —a defect that militates against their use for scientific work.

There is today no desk calculating machine on the market that multiplies directly rather than by repeated addition. Early in the century we had the Swiss Millionaire, but it is no longer manufactured. For each digit of the multiplier it made two rather slow internal strokes—one for the tens and one for the units of the partial products. It could not compete today

in speed with the Marchant. Moreover it was not convenient for division, as each digit of the quotient had to be estimated before the machine was operated; errors, which could easily arise in borderline cases, were troublesome to correct.

We have now achieved satisfactory speed in multiplication. In future developments of commercial machines we may look, not for electronic speeds, but for the removal of the bottleneck of recording results. Not only is this process time-consuming, but it is a source of error. In the late 1920's there appeared the United machine, with separate keyboards for multiplier and multiplicand, and short-cut multiplication. In three seconds a 6×6 product could be formed and printed, as well as its two factors. Unfortunately it did not survive the depression that followed its appearance, and is today only to be seen in a Powers multiplying punch, so that the need for an office desk machine that computes and prints is still with us.

The ease with which modern machines can form sums of products has led, in U. S. A. at any rate, to a revival of the Lagrange interpolation formula.

f_{-1}			
	$\Delta_{-\frac{1}{2}}'$		
f_0		Δ_0''	
	$\Delta_{\frac{1}{2}}'$		$\Delta_{\frac{1}{2}}'''$
f_1		Δ_1''	
	$\Delta_{\frac{3}{2}}'$		
f_2			

Instead of using tabular values and their differences, an interpolate f_n lying a fraction n of the way between f_0 and f_1 is formed by multiplying several tabular values by appropriate weights.

Thus, for a so-called 4-point formula,

$$f_n = L_{-1}^{iv} f_{-1} + L_0^{iv} f_0 + L_1^{iv} f_1 + L_2^{iv} f_2$$

where the L's (whose sum is always 1) are polynomials in n. In 1928 I stated that the principal objections to this formula were that the computer was never sure how many tabular values to use, and was powerless to detect an error in any one of them. These have been largely overcome by duplicating each calculation with different coefficients and tabular values. Thus if there were no discrepancy between two values of f_n, one formed from f_{-1}, f_0 and f_1 and the other from f_0, f_1 and f_2, we could accept them as correct. My objection today is that the method ceases to be convenient if the L's are not directly tabulated; even the most extensive tables available do not have a finer interval than 0.0001. Nevertheless Lagrangean interpolation is a more useful tool today than it was twenty years ago, and lends itself well to the fast tape-controlled special machines now being developed.

At this stage I would like to include an illustration of technique in applying the machines that have been mentioned. It is to inverse interpolation. If $y = f(x)$ is tabulated at equal intervals of x, with its differences, and if we require x corresponding to a given value of y, we must, theoretically, solve an equation of the second, third or fourth order.

Taking Bessel's formula in the form

$$f_n = f_0 + n\Delta_{\frac{1}{2}}' + B''(M_0'' + M_1'') + B'''\Delta_{\frac{1}{2}}'''$$

where

$$B'' = \frac{n(n-1)}{4} \qquad B''' = \frac{n(n-1)(n-\frac{1}{2})}{6}$$

$$M'' = \Delta'' - 0.184\Delta^{iv}$$

(see the description of the throw-back given by Mr. Womersley on page 111f), we may rewrite the formula as

$$f_n - B''(M_0'' + M_1'') - B'''\Delta_{\frac{1}{2}}''' = f_0 + n\Delta_{\frac{1}{2}}'$$

in which n and the B's (which depend on n) are all unknown. If we can find a value of n such that, when substituted in B'' and B''', the two sides of this equation balance, we shall have solved our problem. Two machines are used, one for each side; our object is to balance them. With f_n in the left-hand machine we first ignore the second and third order terms, set f_0 in the right-hand product register and $\Delta_{\frac{1}{2}}'$ on the levers; after balancing the product registers, the right-hand multiplier register shows the n that would result from linear inverse interpolation. This is usually sufficient to determine B''' finally, so the term $-B'''\Delta_{\frac{1}{2}}'''$ is added on the left-hand machine; then $M_0'' + M_1''$ is set on the levers and multiplied by the value of B'', derived (preferably from a critical table) with the approximate value of n. This upsets the balance, which must be restored by turning the right-hand machine; the alteration in n will, in general, alter B'' slightly, and destroy the balance again. This iteration process is continued until there is a static balance with B'' on the left-hand machine, corresponding to n on the right. In systematic work our computers make a rough allowance for the term $B''(M_0'' + M_1'')$ from the beginning, and play a "game" with themselves that the first value of B'' used shall not differ by more than one unit in the last decimal from the final value. Hence inverse interpolation need no longer be shirked, but just taken in the computer's stride. But just a word of warning: do not try this process with automatic machines; hand or semi-automatic machines are needed.

We now turn to a completely different field—that of multi-register machines and their application to problems in finite differences. BABBAGE's efforts more than a century ago to construct a difference engine were frustrated by an enemy that is still with us—apathy and lack of understanding by the administrative lords. However, he inspired SCHEUTZ who succeeded in completing a model—now in a museum in Chicago—of which an English copy, after a brief active life, also found its way to a museum at South Kensington. Other spasmodic attempts produced further museum exhibits. In 1914 T. C. HUDSON, of the Nautical Almanac Office, used a printing Burroughs machine, with repeated settings, for numerical integration. It was on this machine in 1928 that I produced 50,000 7-figure values of (then) new functions for the 4-figure tables compiled in collaboration with MILNE-THOMSON. About this same time we obtained a new Burroughs with an additional register, that could integrate directly from second differences. This machine in 1931 produced 1,200,000 function values, including those of the 8-figure tables of the four principal trigonometrical functions at interval 1″ that were published by PETERS in Berlin in 1939. Copy was also made for a similar 7-figure table; it is now, as far as I know, collecting dust in a Government Department, while less convenient tables, like those of Hof, have been appearing.

This Burroughs triumph was short-lived, for in 1931 the National (really an accounting machine) was "discovered". In its present form it contains six registers, into any combination of which a number can be entered from the keyboard. The contents of any register can be printed, and at the same

time transferred directly to any combination of the remaining registers. It is this direct transfer that is the secret of the success of this machine for scientific work. In the Burroughs, Continental and Sundstrand machines transfer can be effected only through an intermediary known as the cross-footer, and usually requires various idle or spacing strokes; this, in general, trebles the operating time, and is not offset even by the automatic operation sometimes provided—particularly if the latter can only be changed by a service mechanic.

The principal application of the National machine has been to the differencing of functions to their fifth differences, and to summation from sixth finite differences. The former process is used for checking, and for providing differences for interpolation. The latter occurs extensively in subtabulation, or the systematic breaking down of tables to smaller intervals. It is not too much to claim that this machine has altered our whole attitude to table-making, and has induced a willingness to produce tables that would have been considered too extensive or laborious fifteen years ago. The means of checking values at regular intervals of an independent variable by differencing has relieved modern computing teams of much labour and anxiety and leaves little excuse for error. The 2000 errors marked in our copy of HAYASHI's 1926 tables were all found by differencing—and might well have been found in this way by the author himself.

In the work of solving differential equations, multi-register machines go on from where differential analysers leave off. Even the most elaborate and costly analysers can hardly produce more than four significant figures, but there are occasions when more are required. Once we revert to step-by-step summation, machines that will add, subtract, print and store intermediate results come to our aid. It is the storage requirement (for various orders of differences) that calls for many registers—all we have now, and those we may expect in the future. I am confident that the next decade will see a great advance in our treatment of differential equations by numerical integration, and in the evaluation of definite integrals by quadrature.

Finally we come to the punched-card machines, which really fall in the multi-register group. They are distinguished by the fact that they receive their data from cards in which numbers are represented by punched holes; this enables them to work with great speed. When full advantage can be taken of the system, it is possible to add anything up to 100 figures a second. There are two rival machines of this type. One is electrical and in England (and formerly in U. S. A.) is still called, after its original inventor, the HOLLERITH; in America it seems to be called now the IBM, after its manufacturing company, International Business Machines. The other machine is mechanical, named after its inventor, POWERS. For scientific work the flexibility given by easily-changed electrical connections has been in favour of the Hollerith; the Powers, although well established as an accounting machine, has made little inroad into technical computing.

There are several members of the Hollerith family. Besides machines for punching the holes in cards and checking and reading them, there is a sorter which arranges cards in numerical order. The main machine is the tabulator, with, say, six or eight adding mechanisms; it can print the contents of cards, add and subtract them in groups, and print the totals. The English Hollerith can transfer a number in any register positively or nega-

tively to any combination of the other registers, but will not subtract directly from a card—a serious drawback. The IBM, on the other hand, will subtract from a card, but is less developed in its transfer facilities. Neither machine is, unfortunately, above reproach in reliability or accuracy—a point to which their designing engineers could well afford to give more attention.

The first scientific application of the Hollerith—in 1928—was to the summation of harmonic terms, or, in other words, Fourier synthesis. In this way the principal terms in the motion of the Moon from 1935 to 2000 were computed. The ordinates of the various harmonics to be compounded were taken from E. W. BROWN's *Tables of the Moon*, and punched on to half a million cards. These were brought together in ever-differing groups; because the periods of the different harmonics are incommensurable, no card ever had the same partners twice, although it may have been used 20 times in the 65 years covered. Something like 100 million figures were added in groups and group totals printed in the course of seven months. I showed this to Brown in the summer of 1928; he had done a great deal of this synthesis himself by hand, and I shall ever remember his ecstasies of rapture as he saw his figures being added at the rate of 20 or 30 a second. I think I am right in saying that the enthusiasm with which he described the process on his return first stimulated W. J. ECKERT, the leading American pioneer of these machines for scientific calculations.

In 1932, when the Hollerith first introduced inter-register transfer, it became a rival to the then newly-discovered National for differencing and integrating. That it has not been used more (till recently) for this type of work is probably due to the fact that its rental is such that it must not be allowed any idle moments; its working speed is so great that there may easily be difficulty in producing a continuous supply of work for it. In 1944 we used it to subtabulate to fifths in each direction a large 5×5 grid of ballistic tables that we had prepared for the U. S. Air Forces. The million figures in the final results were so beautifully printed that photostatic copies met all requirements. It may be noted that whereas the National builds up one difference at a time, the Hollerith can take advantage of the method proposed by Babbage and incorporated by SCHEUTZ—that of integrating in two strokes, in one of which even differences are added to odd, while in the other odd differences are added to even. It is not difficult to produce 1000 values an hour by summation; this is several times as fast as a National, but is not necessarily more economical, as the annual rental of a Hollerith is comparable with the initial purchase price of a National.

Another Hollerith machine—the multiplying punch—can sense numbers a, b, c, d on a card, form combinations like $ab + c \pm d$, and punch them on the same card. In 1933 this was used for extensive conversion of spherical to rectangular co-ordinates. Its use was also suggested by the author for forming apparent places of stars from equations of the type

$$\alpha = \alpha_0 + Aa + Bb + Cc + Dd$$

where the capital letters represent quantities that are constant for all stars for any one date, and the small letters represent quantities that are constant for any one star for all dates. At that time this work was divided between four countries, none of which could justify a Hollerith installation,

but war conditions have led to the application of this method more recently. In preparing *The 1940 Heat Drop Tables* 2000 initial quantities were first punched on cards, and from these, with the aid of reproducer punches, the multiplier punch, the sorter and the tabulator 66,000 values of the heat drop were produced in a few weeks in the form of copy for press.

Towards the end of the war an IBM machine was employed in England on Fourier synthesis. In the course of the determination of crystal structure by X-ray analysis, it becomes necessary to synthesise the electron density over the volume of a unit cell from expressions of the type

$$\rho = \sum_{h=0}^{h=H} \sum_{k=0}^{k=K} \sum_{l=0}^{l=L} F_{hkl} \cos 2\pi hx \cos 2\pi ky \cos 2\pi lz$$

where H, K and L are integers not exceeding 40 (10, 20, 35 could be considered representative), F_{hkl} are coefficients derived from observation and x, y, z are arbitrary uniform divisions of the cell edges. If in a typical case x and y extend to 30 and z to 60, we may require to evaluate this expression for $30 \times 30 \times 60 = 54{,}000$ points, each of which involves many multiplications and additions; the labour of hand work becomes prohibitive. Various transformations suggested by BEEVERS and LIPSON reduce the work considerably, and their scheme has now been mechanised by having a stack of "master" cards, from which those required for any synthesis are selected (mechanically), reproduced[1] (again mechanically) on to "slave" or "detail" cards, which are added, and the results printed. The master cards are returned to stock for future use. The cards extend only to the point where $2\pi x = \frac{1}{2}\pi$; by slight changes of plugging four runs extend the range to 2π. Provision is made for an interval of 1/120 in x. The successful work still being done on salts of penicillin leads to the hope that a complete bureau will be established to relieve crystallographers of their heaviest burden.

The solution of simultaneous linear equations is so laborious that MALLOCK and WILBUR have each made machines for the purpose. Mallock's machine (only one has been made) is in use by the Mathematical Laboratory of Cambridge University; Wilbur's is in storage at M.I.T. What is needed is a machine that is not limited to about ten unknowns, as up to that point the labour of using electric automatic machines is not excessive. A recent study by H. O. HARTLEY (not yet published) of the problem of many unknowns has led to a Hollerith tabulator (*not* multiplying punch) technique that has proved successful in its early trials. Success in being able to solve large numbers of simultaneous equations readily, whether by an existing machine or otherwise, will confer a great boon on mathematicians, and enable new fields to be explored.

I am convinced that the day of the desk machine is not yet over, or even threatened by the half dozen or so large and special machines that have come into being during the war. Nevertheless I join with others in admiring these machines, and, after seeing so much binary multiplication, feel that LEWIS CARROL should be alive now to write *Alice in Onederland*. There is however much more to be done before the usefulness to science of the commercial machine is exhausted. There is still room for development in application by Scientific Computing Service, the National Physical Laboratory, the M.I.T. Centre of Analysis, the Watson Computing Labora-

tory, the New York Mathematical Tables Project, the U. S. Naval Observatory, and others. Let us hear from each of those some report of their ". . . other aids to computation."

L. J. C.

[1] My youthful typist once typed "mater" cards; perhaps she thought they were better fitted for reproducing.

PUBLICATIONS OF THE AUTHOR

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"On the application of the Brunsviga-Dupla calculating machine to double summation with finite differences," R.A.S., *Mo. Notices*, v. 88, 1928, p. 447–459.

"On the construction of tables by interpolation," R.A.S., *Mo. Notices*, v. 88, 1928, p. 506–523.

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"The Nautical Almanac Office Burroughs machine," R.A.S., *Mo. Notices*, v. 92, 1932, p. 523–541.

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The Hollerith and Powers Tabulating Machines, Printed for private circulation, London, 1933, 48 p.

Articles "Adding Machines" and "Calculating Machines" in Hutchinson's *Technical and Scientific Encyclopaedia*, London, 1934.

"Inverse interpolation" and "Scientific applications of the National Accounting Machine," R. Statistical So., *J.*, v. 3, 1936, suppl., p. 87–114.

"Interpolation and allied tables"; reprinted from the *Nautical Almanac* for 1937, p. 784–809, 839, 926–943. Published by H.M. Stationery Office.

"The application of the Brunsviga Twin 13Z calculating machine to the Hartmann formula for the reduction of prismatic spectograms," *The Observatory*, v. 60, 1937, p. 70–73.

(With G. B. HEY and H. G. HUDSON.) "The application of Hollerith equipment to an agricultural investigation," R. Statistical So., *J.*, v. 4, 1937, suppl., p. 210–224.

On the application of the Brunsviga Twin 13Z calculating machine to artillery survey, London, Scientific Computing Service, 1938, 18 p.

"Calculating Machines," an Appendix to L. R. CONNOR, *Statistics in Theory and Practice*, third ed., London, Pitman, 1938, p. 349–371.

"The use of calculating machines in ray tracing," Phys. So., *Proc.*, v. 52, 1940, p. 246–252.

The Twin Marchant Calculating Machine and its Application to Survey Problems, London, Scientific Computing Service, 1942, 40 p.

"Mechanical computing," Appendix I to DAVID CLARK, *Plane and Geodetic Surveying*, third rev. ed., v. 2, London, 1944, p. 462–473.

"Careers for girls," *Math. Gazette*, v. 28, 1944, p. 90–95.

"Recent progress in scientific computing," *J. Scient. Instruments*, v. 21, 1944, p. 129–135.

RECENT MATHEMATICAL TABLES

316[A–D].—FRANKLIN MARION TURRELL, *Tables of Surfaces and Volumes of Spheres and of Prolate and Oblate Spheroids, and Spheroidal Coefficients.* Berkeley and Los Angeles, University of California Press, 1946. xxxiv, 153 p. Offset print. 13.9 × 21.6 cm. $2.00.

The author of this work is an assistant plant physiologist at the Citrus Experiment Station of the University of California. He tells us that it is necessary to find the areas and volumes of such fruits as lemons, grapefruit, melons, nuts, etc. The lemon is regarded as approximating to a prolate spheroid, and a grapefruit to an oblate spheroid. Assuming that $2a$ and $2b$ are the axes of the ellipse of revolution and $\delta = 2a - 2b$, $\rho = 2b/2a$, e = eccentricity, there are the following tables:

Table 1 (p. 1–3): Surfaces and Volumes of Spheres, $\delta = 0$, d (diameter) $= 1(.1)15$, 3S up to $d = 5.6$, then integral values to the end.

Table 2 (p. 4–133): Surfaces and Volumes of Spheroids, for $2a = N(.1)15$, to 3S or to the nearest integer, last figure uncertain. $N = 1$, $\delta = .1(.1).5$; $N = 1.5$, $\delta = .6, .7$; $N = 2$, $\delta = .8(.1)1$; $N = 2.5$, $\delta = 1.1(.1)1.5$; $N = 3$, $\delta = 1.6(.1)2$; $N = 5$, $\delta = 2.1(.1)2.5$; $N = 7$, $\delta = 2.6$; $N = 7.5$, $\delta = 2.7(.1)3$. The smallest value for $2a$ is 1 and for $2b$ is .5. ρ varies from about .33 to 1. A centimeter is the linear unit used. These values were calculated from the formulae

$$\begin{aligned} &\text{Sphere, } S = \pi d^2,\ V = (1/6)\pi d^3; \\ &\text{Prolate spheroid, } S = 2\pi b^2 + 2\pi(ab/e)\sin^{-1} e, \\ &\qquad V = (4/3)\pi ab^2; \\ &\text{Oblate spheroid, } S = 2\pi a^2 + \pi(b^2/e)\ln[(1+e)/(1-e)], \\ &\qquad V = (4/3)\pi a^2 b. \end{aligned}$$

Table 1 was checked by Tables 21–22, p. 87–91 of *Chemical Engineers' Handbook*, ed. by J. H. PERRY, second ed., New York, 1941. In T. 21 S and V are given for

$$d = [\tfrac{1}{32}(\tfrac{1}{32})\tfrac{1}{2}(\tfrac{1}{16})4(\tfrac{1}{8})12(\tfrac{1}{4})34\tfrac{1}{2}(\tfrac{1}{2})100;\ 5S \text{ or } 6S].$$

In T. 22, V is given for $d = [1(.01)10; 4S]$. In F. G. GAUSS, *Fünfstellige vollständige Logarithmische u. trigonometrische Tafeln*, Berlin, 1870, p. 130, there is a table for S and V for r (radius) $= 0(1)100$ to the nearest integer. This same table is given in Hamburg, Sternwarte, *Sammlung von Hilfstafeln, A–F*, Hamburg, 1916, p. A31.

There are also the 20-place tables for the area and volume of a sphere by J. MODER, 1886; see *MTAC*, v. 2, p. 87.

In the case of **Table 2**, the 4-place tables with 3-place arguments of H. B. DWIGHT (*Mathematical Tables* . . . , third impression with additions, 1944, *MTAC*, v. 1, p. 180, p. 114–115) were used for evaluating $\sin^{-1} e$. In other elements the calculations were to 3D or 4D, and the final values of the formulae rounded off to 3S. The values in Tables 1–2 were further checked by considering the series of values in the ratios S_{n-1}/S_n, V_{n-1}/V_n, $n = 1(1)n$. "If the ratios of adjacent values of S or V changed considerably from one ratio to another, the solutions for the pair of arguments concerned were recalculated. If the error was not discovered, recalculation was made of S or V as required for other adjacent pairs of arguments."

Tests of the accuracy of Tables 1–2, when used for estimating surfaces and volumes of citrus fruit, were made by F. M. TURRELL, JANE P. CARLSON & L. J. KLOTZ (Amer. So. Horticultural Sci., *Proc.*, v. 46, 1945, p. 159–165). They found that the mean error between actual surface measurements and tabular values ranged from 2.23% to 5.59% for different kinds of fruit. In the case of volume measurements and tabular values the mean error ranged from 1.95% to 8.35% for different kinds of fruit. Interpolation for Tables 1–2 is discussed and illustrated by examples.

Tables 3–6 (p. 134–137), spheroidal coefficients for extending **Table 2**, were prepared by F. M. TURRELL and A. P. VANSELOW. Their characteristics, methods of calculation, methods of checking, and errors introduced by their use are discussed in Amer. So. Hort. Sci., *Proc.*, v. 47, 1946.

Many different computers are listed as having been connected with the preparation of the tables discussed above.

R. C. A.

317[A, D].—M. J. BUERGER & GILBERT E. KLEIN, "Correction of diffraction amplitudes for Lorentz and polarization factors," *J. Appl. Physics*, v. 17, Apr. 1946, p. 285–306. 20 × 26.7 cm.

The following 12 tables are given:

T. I–II, VI–VII. $y_1 = \frac{1}{2}(1 + \cos^2 2x)$, and $y_1^{\frac{1}{2}}$ for $\sin x = [0(.001).999; 4D]$.

I = II = y_1 are really identical tables; also VI = VII = $y_1^{\frac{1}{2}}$. It seems rather extraordinary that in a paper emanating from the Massachusetts Institute of Technology it should not have been recognized that tables for parameters $\sin x = 0(.001).999$ and $\sigma = 2 \sin x = 0(.002)1.998$ are identical.

T. III, VIII. $y_2 = \csc x$, and $y_2^{\frac{1}{2}}$ for $x = [0(0^\circ.1)179^\circ.9$; 4–5S].
T. IV–V, IX–X. $y_3 = (1 + \cos^2 2x)/[2 \sin 2x]$, and $y_3^{\frac{1}{2}}$ for $\sin x = 0(.001).999$; mostly 4D]. IV = V = y_3; IX = X = $y_3^{\frac{1}{2}}$.
T. XI. $(h^2 + k^2)^{\frac{1}{2}}$, $h = 0(1)30$, $k = 0(1)30$, to 4S.
T. XII. $(h^2 + k^2 + hk)^{\frac{1}{2}}$, $h = 0(1)30$, $k = 0(1)30$, to 4S.

Tables for $h^2 + k^2$ and for $h^2 + k^2 + hk$ were published previously by the authors in *J. Appl. Physics*, v. 16, p. 412; in this same article are also the two tables of $1/y_3$ (p. 414, 416), and also two of $1/y_1$ (p. 415, 417). Compare *MTAC*, v. 1, p. 436.

318[C, D].—CARL THEODOR ALBRECHT [1843–1915], *Logarithmic and Trigonometric Tables to Five Decimal Places*. New York, G. E. Stechert & Co., 1946 [the title-page gives "1944" incorrectly; the present edition is a reprint of the new 1944 edition]. vi, 147 p. 15.3 × 22.8 cm. Photolithographed by The Murray Printing Co., Cambridge, Mass. $1.50.

This volume contains the following: a preface by Stechert (p. iii–iv); **Table I**, Logarithms of Numbers (1000–10000, p. 1–37); conversion table for natural and common logarithms (p. 38); **Table II**, Logarithms of sines and tangents, 0(1″) 3° (p. 39–75); conversion table for degrees measure and radian measure (p. 76); **Table III**, Logarithms of [the six] trigonometric functions at interval 1′ (p. 77–122); **Table IV**, Natural trigonometric functions of sin, tan, cot, cos, at interval 1′ (p. 123–146, reprinted from E. R. HEDRICK, *Logarithmic and Trigonometric Tables*); Constants (p. 147).

The publishers tell us that the Albrecht Tables in their original form had been used at the Drexel Institute of Technology, Philadelphia, for the past 25 years and that Institute's department of mathematics had suggested the form of the 1944 edition. Albrecht's work appeared first with the title *Logarithmisch-Trigonometrische Tafeln mit fünf Decimalstellen*, Berlin, 1884, xvi, 172 p. The nineteenth ed. was published at Stuttgart in 1930. It seems to have been an offset print of this latter edition which Stechert had made in 1932 by Henri Dupuy, Paris, and distributed in New York; xvi, 176 p. Apparently this edition differs very slightly from the original edition (which we have not seen). Pages 6–126 are the same as p. 1–122 of the volume under review. Then follow a table for addition and subtraction logarithms (p. 127–148), and other miscellaneous tables, formulae and constants (p. 149–176).

For many years, after 1873, Albrecht was a "Sectionschef" of the Prussian Geodetic Institute. In 1883 he edited the tenth edition of *Logarithmisch-Trigonometrische Tafeln mit sechs Decimalstellen* by CARL BREMIKER (1804–1877), of which there was an eleventh stereotyped ed., published in Berlin in 1890. The English edition (London, 1887), with new material by ALFRED LODGE, was therefore adapted from the Albrecht-Bremiker work.

R. C. A.

319[C, D, E].—GEORG VEGA (1756–1802). **1.** *Thesaurus Logarithmorum Completus, ex Arithmetica Logarithmica,* et ex *Trigonometria Artificiali Adriani Vlacci collectus, plurimis erroribus purgatus, in novum ordinem redactus,* . . . WOLFRAMII *denique tabula logarithmorum naturalium locupletatus a Georgio Vega* . . . , Leipzig, 1794 [also with a briefer German title page]. viii, XXX, 685 p. Size of the Brown University copy 20.5 × 33 cm. P. I–XXX, both Latin Introductio and German Einleitung in parallel columns.

6. Facsimile, *Georg Vega 10 Place Logarithms including Wolfram's Tables of Natural Logarithms. Reprint of the rare edition of* 1794. New York, G. E. Stechert & Co., 31 East Tenth St., New York City, 1946. viii, XXXI, 684 p. Photolithographed by The Murray Printing Co., Cambridge, Mass. 15 × 23 cm. $7.50.

GEORG *Baron* VON VEGA (1756–1802) was an Austrian Army Officer who taught mathematics and was the author of several mathematical works, including tables. His *Thesaurus* of 1794 contains the following tables: T. I (p. 1–310), log N, N = [1(1)100999; 10D], 1(1)1000 without Δ, but the rest with Δ. T. II (p. 311–629), log sines, cosines, tangents, and cotangents, for [0(1″)2°; 10D] and the rest of the quadrant to 88°, at interval 10″, with Δ. This is followed (p. 631–632) by the natural sines for [0(1″)12′; 12D]. The appendix (p. 633–685) includes (p. 634–635) the circular measure of 1°(1°)360°, 1′(1′)60′, 1″(1″)60″, all to 11D; and a small table expressing minutes and seconds as fractions of a degree. P. 641–684 contains a reprint of a remarkable table which took the Dutch artillery officer, J. WOLFRAM, six years to compute, hyperbolic logarithms, that is, T. III: ln N, for N = [1(1)2200 (primes and a few others) 10009; 48D]. These were first published in J. C. SCHULZE, *Recueil de Tables Logarithmiques, Trigonométriques et autres nécessaires dans les Mathématiques Pratiques*, also with German t. p., Berlin, v. 1, 1778, p. 189–259 and in *Astron. Jahrbuch für das Jahr* 1783, Berlin, 1780, part 2, p. 191. In this latter place Wolfram first gave the values for N = 9769, 9781, 9787, 9871, 9883, 9907. Since Vega's value for N = 9883 is in error by a unit in the forty-eighth decimal place,[7] it has been surmised that Vega calculated these six results independently. There are at least 31 common errors in Wolfram and Vega namely: for N = 390, 829, 1087 (2), 1099, 1409, 1900, 1937, 1938, 2022, 2064, 2093, 2173, 2174, 2175, 3481, 3571, 3763, 3967, 4033, 4321, 4757, 5123, 6343, 7247, 7853, 8023, 8837, 8963, 9409, 9623. These are discussed by DUARTE,[1] GRAY,[2] GUDERMANN,[3] KULIK,[4] STEINHAUSER,[5] WACKERBARTH,[6] and in my notes in *Scripta Mathematica*.[7] Another error in Wolfram, N = 4891, was correct in Vega.[7] C. R. COSENS noted the four last-figure unit errors calling for increases in 2173, 2175, and decreases in 1087, 2174.

Tables I–II of Vega are mainly reprints of other works, the first being of A. VLACQ, *Arithmetica Logarithmica* . . . , Gouda, 1628, log N, for N = [1(1)100000; 10D]. Log N for N = [100001(1)100999; 10D], appearing for the first time in Briggs, *Arithmetica Logarithmica*, 1624, to 14D (*MTAC*, v. 1, p. 170; v. 2, p. 94), contrary to what Glaisher[13] 1873 states, p. 139, were calculated by "Lieut. Dorfmund." Table II is not reprinted entirely from Vlacq's *Trigonometria Artificialis* . . . , Gouda, 1633, since the logarithms for the first two degrees were calculated for the work by DORFMUND. Vega took great pains to make his T. I accurate and remarkably succeeded.

Vega's *Thesaurus* was reviewed by GAUSS[8] who found T. I exceedingly accurate, but estimated that in T. II, the trigonometric canon, there were from 31983 to 47746 last-figure (tenth decimal) errors, most of them only amounting to a unit, but some to as much as 3 or 4 units. This table has been discussed at length by GLAISHER[9] and lists of errors were published by GRONAU,[10] M. VON LEBER[11] (who arrived at a result similar to that of Gauss), and HOBERT & IDELER.[12] An abridgment of T. II to 10D, at interval 10′, taking account of all known errors at the time of publication by the Astronomische Gesellschaft, is in C. BÖRGEN, *Logarithmisch-Trigonometrische Tafel auf 11 (bezw. 10) Stellen*, Leipzig, 1908, p. 29–34.

It is also of importance to refer to Vega's list of 120 Errata in the original *Thesaurus*. The first page is at the end of the introduction, p. XXX, in Latin, and the second is on p. 685, in Latin and German; the contents of these two lists are identical. The second sheet was for cutting up so that the correct figures might be pasted over those which were erroneous. In some copies the list is much more complete than in others; "120" appears to be the maximum number of entries.

Glaisher notes[13] "There is a great difference in the appearance of different copies of the work. In some the tables are beautifully printed on thick white paper, with wide margin, so that the book forms one of the handsomest collections of tables we have seen; while in others the paper is thin and discoloured; all are printed from the same type."

Of Vega's *Thesaurus* there have been five reprints, the first two, at least, at Florence, by the Istituto Geografico Militare, in facsimile folio format. These were photozincographic reproductions, and the first,

2. was got out in 1889, in an edition of 250 copies, in order to meet the needs of the geodetic service. It was distributed to all libraries, astronomical observatories, and other scientific

institutions in Italy, and was presented at the triennial meeting of the International Geodetic Association in Paris in 1889. This edition was soon exhausted, contains xxx, 684 p. 22 × 33.2 cm. (the copy consulted may have been trimmed while being bound). 121 entries (the last wrirten by hand, for ln 1099) are given on the full Errata sheet, p. xxx, and all of these except 1 were corrected in the text before reproduction. On the half-title is pasted a piece of paper (10.3 × 7.8 cm.) with the legend:

Riproduzione fotozincografica
Dell' Istituto Geografico Militare
Firenze, 1889.

The work was apparently issued with paper covers of colored paper, on which the half-title, without border, was printed, and a similar piece of paper pasted. There is a copy of this edition at Harvard University. Since the original plates had been preserved the Istituto printed, on thicker paper,

3. Another edition of 200 copies, in 1896. 24 × 34.1 cm. In addition to the corrections made in edition **2** the following were also made in **3** before reproduction:

p. 172. The first three figures, 777, which are to be found corresponding to number 59840 were lowered by one line, and the proper asterisks affixed to the numbers in columns 2, 3, 4.
355. log tan 1° 26′ 12″, *for* 4249, *read* 4149 (Westphal),
log cot 1° 26′ 12″, *for* 5751, *read* 5851 (Westphal).
414. log cot 9° 5′ 50″, *for* 7008, *read* 6908 (Luther).
679. ln 6343, *for* 1623, *read* 1633.

The print facsimile is slightly larger than the original, and the generous margins and good paper add to the attractions of the volume. What were thought of as pages i–iii of the original edition are used for "Prefazione alla seconda edizione fotozincografica." There seem to have been changes in the course of printing of this edition since the last paragraph of the "prefazione" is different in two different copies, **3A** and **3B**, at Brown University.

In his *Nouvelles Tables Logarithmiques à 36 décimales*, 1933, DUARTE seems to state (p. xxiv), perhaps copying H. ANDOYER, *Nouvelles Tables Trigonométriques Fondamentales* (*Logarithmes*), Paris, 1911 (p. vi), that the Istituto published

4. A third edition in 1910, presumably at Florence. The *Jahrb. ü. d. Fortschritte d. Math.* for 1910 simply lists an edition of the *Thesaurus* published at Milan. I have not seen a copy of this edition. Andoyer's remark concerning the Florentine editions ("Mais les erreurs de l'édition originale n'ont pas été corrigées"), copied by Duarte, is both misleading and incorrect; misleading, because all but one of the corrections of Vega's Errata sheet were made in the text of editions **2**, **3**, **4**; and incorrect, because other corrections were made in **3** and **4**.

5. The next edition was made in Vienna by G. E. Stechert & Co., New York, in 1923, and is a slightly enlarged print facsimile. 21.5 × 33.7 cm. The half-title of the original edition was replaced by a new title page: *Georg Vega. 10 Place Logarithms including Wolfram's Tables of Natural Logarithms. Reprint of the Rare Edition of 1794.* The Errata sheet is the same as the printed part of the list in **3**, and all of these corrections were made in the text before reproduction of the original edition. Hence Henderson's statements "All the errors of the original appear" (art. "Mathematical Tables," *Encycl. Britannica*, 14th ed., 1929, and *Bibliotheca Tabularum, Mathematicarum*, 1926, p. 162) are misleading and raise a doubt in one's mind as to whether Henderson thought that the presence of the Errata sheet in **5** meant that the errata in the text had not been corrected, or, that he intended his remark to apply only to errors not on Vega's Errata sheet. What was p. 685 is unnumbered and follows p. XXX. This edition was sold for $12.50.

6. Since the unsold stock of **5** was stored in Leipzig and destroyed during the recent war, G. E. Stechert & Co. decided to make another reprint at a time when paper and binding material shortages were acute. Hence the miniature photolithographed copy under review. The size of the new print page as compared with the original, **1**, is about as 34:61. The

new margins are very narrow, but the print is clear, the type is adequately large for use, and the volume is much easier to handle than the large folio.

During the past 150 years various errata lists for Vega's *Thesaurus*, besides those already mentioned, have been published.[14] So far as T. I is concerned Peters believed that he listed all of Vega's 303 errors (301 unit errors, and 2 of 2 units in the tenth decimal place) on p. xvi of his work, 1922, but he omitted the unit error in 100330, noted by Lefort. Peters' list, mainly a correction and expansion of Lefort's lists,[15] almost wholly duplicated by Leber,[11] is reprinted, with extra information about the entries, in JAMES HENDERSON, *Bibliotheca Tabularum Mathematicarum*, Cambridge, 1926, p. 94–95.

The reader naturally inquires, Has Vega's work ever been superseded? The answer is, partially. T. I [N = 1(1)100 000] and T. III N = [1(1)146 (primes) 9973] by J. T. PETERS, *Zehnstellige Logarithmentafel*, v. 1: *Zehnstellige Logarithmen der Zahlen von* 1 *bis* 100 000, Berlin, 1922, and an auxiliary table, [v. 3, 1919], showing corrections for second differences; T. III with 10 errors (N = 829, 1087, 1409, 3967, 6343, 7247, 8837, 8963, 9623, 9883), is in the Appendix. Two of Vega's errors (in 3571, 7853) are here corrected. There is a table log N, N = [10 000(1)100 000; 20D], δ^{2a} by A. J. THOMPSON, *Logarithmetica Britannica* . . . , 9 parts, Cambridge, 1924–1937; the final published part, no. 2, N = 20 000–30 000, is in the press. But for T. II no 10D table covering Vega's range has been published since 1794. Andoyer's *Nouvelles Tables Trigonométriques Fondamentales* (*Logarithmes*), Paris, 1911, contains a 14D table, at interval 10″, which is as complete, in interval, for 3°–88°. But these volumes are expensive, and the first one is excessively difficult, if not impossible, to secure, except in a library. Since the Vega volume is reasonably accurate it still fills a decided need.

No comprehensive review of editions of Vega should fail to refer to a ten-place logarithm table of numbers 10000(1)100009, Δ, issued with author's name, W. W. DUFFIELD, then superintendent of the U. S. Coast and Geodetic Survey, in its *Report for 1896*, Washington, 1897, appendix, no. 12, "Logarithms, their nature, computation, and uses, with logarithmic tables of numbers and circular functions to ten places of decimals—Part I," p. 395–722 (the table occupying p. 422–721. 22.8 × 38.5 cm.). On p. 397 we find the following:

"In the accompanying logarithmic tables all the mantissae have been computed to twelve places of decimals, and whenever the eleventh and twelfth places exceeded 50 the tenth place has been increased by unity, or 1; but whenever the eleventh and twelfth places were 50 or less than that number the tenth place has not been increased."

"When these computations were begun I was not aware that Baron George von Vega had preceded me in his Thesaurus Logarithmorum Completus. But my own results have been carefully compared with those of Von Vega, and whenever any difference was detected the computation was made anew. In this way many serious errors (undoubtedly typographical) in Baron von Vega's tables have been discovered and corrected."

The gross falsity of the superintendent's statement concerning computations of his staff (to which no reference is made) was the basis of much comment on their part. With devastating completeness Peters showed that the Duffield staff carried out computations as stated to only about 26004, but thereafter simply copied Vega's table. The explanation of the three later numbers which Duffield had correct (31653, 38051 and 60704), while Vega was incorrect, is that Duffield had his computers carry through the work for a few later random values. There is a one-to-one correspondence between the 267 other errors in Vega and Duffield. Up to 26 004, the 1897 work had corrected 33 Vega errors. See also Henderson, *l.c.* WILLIAM WARD DUFFIELD (1823–1907) served on the staff of General Pillow in the Mexican War, 1847–48, and during the Civil War, 1861–65, commanded the 4th Michigan infantry. He was brevetted major-general in 1863, elected state senator for Michigan in 1878 and appointed chief engineer for railways in Michigan, New York, Illinois and Texas; and U. S. engineer of improvements on Wabash and White rivers in 1892. He was superintendent of the U. S. C. G. S. 1894–98 (*Encycl. Americana*). From a scientific genius, BENJAMIN PEIRCE, superintendent 1867–74, to a scientific pigmy!

R. C. A.

[1] F. J. DUARTE, (a) *Nouvelles Tables de Log n! à 33 décimales depuis n = 1 jusqu'à n = 3000*, Paris, 1927, p. III; (b) *Nouvelles Tables Logarithmiques à 36 Décimales*, Paris, 1933, p. xxii.

[2] P. GRAY, *Tables for the Formation of Logarithms & Antilogarithms*, London, 1865, p. 39.

[3] C. GUDERMANN, *J. f. d. reine u. angew. Math.*, v. 9, 1832, p. 362.

[4] J. P. KULIK, *Astron. Nach.*, v. 3, 1825, cols. 191–192.

[5] A. STEINHAUSER, *Hilfstafeln zur präcisen Berechnung zwanzigstelliger Logarithmen* . . ., Vienna, 1880, p. 1.

[6] A. F. D. WACKERBARTH, R.A.S., *Mo. No.*, v. 27, 1867, p. 254.

[7] R. C. A., *Scripta Math.*, v. 4, 1936, p. 99, 293. See also *MTAC*, v. 1, p. 57.

[8] C. F. GAUSS, "Einige Bemerkungen zu Vega's *Thesaurus Logarithmorum*," *Astron. Nachr.*, v. 32, 1851, cols. 181–187; also in his *Werke*, v. 3, 1866 and 1876, p. 257–264.

[9] J. W. L. GLAISHER, "On logarithmic tables," R.A.S., *Mo. No.*, v. 33, 1873, p. 440–451; see also an appended letter of J. N. LEWIS. See also v. 32, 1872, p. 288–290, and v. 34, 1874, p. 471–475.

[10] J. F. W. GRONAU, "Tafeln für die hyperbolischen Sectoren und für die Logarithmen ihrer Sinus und Cosinus," Natur. Ges., Danzig, *Neueste Schriften*, v. 6, no. 4, 1862, p. vi. Lists 99 errors in T. II.

[11] M. VON LEBER, *Tabularum ad Faciliorem et Breviorem, in Georgii Vegae "Thesauri Logarithmorum" magnis Canonibus, Interpolationis Computationem utilium, Trias*, Vienna, 1897. He lists 272 errors in the tenth place of T. I, of which all but five are unit errors; the three serious ones had been corrected by Vega himself. Peters showed that 8 other entries by Leber as errors in Vega, were in fact correct. Also 2148 errors in T. II.

[12] J. P. HOBERT & L. IDELER, *Neue Trigonometrische Tafeln für die Decimaleintheilung des Quadranten*, Berlin, 1799, p. 350–351. There are here 168 corrections of T. II, 157 final-unit errors, 10 2-unit errors, 1 3-unit error.

[13] J. W. L. GLAISHER, B.A.A.S., *Report, 1873*, p. 138.

[14] Other lists are as follows:
K. KNORRE, *Astr. Nach.*, v. 7, 1829, col. 62. Error in T. I.
R. LUTHER, *Astr. Nach.*, v. 44, 1856, cols. 239–240. Error in T. II.
E. SANG, R. So. Edinburgh, *Proc.*, v. 8, 1875, p. 376. 40 unit errors, and one error listed by Vega, in T. I, N = 20071–29703.
D. J. M. M'KENZIE, *Bull. Sci. Math.*, s. 2, v. 4, 1880, p. 31f. Error in T. III.
A. WESTPHAL, *Astr. Nach.*, v. 114, 1886, cols. 333–334. Error in T. II.
J. FRISCHAUF, *Astr. Nach.*, v. 174, 1907, col. 173. 2 errors in T. I.
G. WITT, *Astr. Nach.*, v. 178, 1908, cols. 263–266. 23 errors in T. II.
P. ADRIAN, *Astr. Nach.*, v. 198, 1914, cols. 167f, 327f. Errors in T. I.

[15] F. LEFORT, Paris, Observatoire, *Annales, Mémoires*, v. 4, 1858, p. [148]–[150]. 300 errors listed in T. I. Peters' table does not adopt 7 of the final-digit unit changes demanded by Lefort for 26188, 29163, 30499, 31735, 34162, 34358, 60096. There are 25 very serious errors in T. I of 1, listed by Lefort, but all of these are in Vega's Errata list.

F. LEFORT, R. So. Edinburgh, *Proc.*, v. 8, 1875, p. 571–574, 587; also by E. SANG, p. 586–587. Lefort lists 287 last-figure errors (all except 5, one unit in the tenth decimal place) in T. I, 2 in Table II, and 2 in T. III (1099, 7853). There is a duplication of statement of 7 correct logarithms as erroneous. But furthermore, Lefort notes 6 errors, all in Vega's list of Errata, which implies inclusion of 100330 since the unit error for 10033 is listed, but it is not actually stated, as in Lefort.

320[D].—BENGT STRÖMGREN (1908–), *Optical Sine-Tables giving seven-figure values of $x - \sin x$ with arguments x and* sin x. Geodaetisk Institut, Copenhagen, *Skrifter*, s. 3, v. 5, 1945, 63 p. 22.8 × 28.6 cm.

In the fields of optics and astronomy, it has long been realized that tables giving the quantity $x - \sin x$ are useful in numerical calculations of the properties of optical systems. Some previously published tables are as follows:

1. A. M. LEGENDRE, *Exercices de Calcul Intégral*, v. 3, 1816, p. 178. Table of $\frac{1}{2}(2x - \sin 2x)$ for $x = [0(1°)90°; 10D]$, Δ^3.
2. J. F. ENCKE, "Ueber die Berechnung der Bahnen der Doppelsterne," *Berliner Astron. Jahrb. f. 1832*, 1830, p. 297–304; also in J. F. ENCKE, *Ges. mathem. u. astron. Abhandlungen*, v. 3, Berlin, 1889, p. 71–78, $2x - \sin 2x$, for $x = [0(10')90°; 5D]$, Δ.
3. A. STEINHEIL & E. VOIT, *Handbuch der angewandten Optik*, v. 1, Leipzig, 1891, p. 271–314, $x^0 - (\sin x)^r$, $x = [0(10'')2° 46'40''; 0''.001]$, $[2° 46' 50''(10'')30°; 0''.01]$. This table is not included in the English translation, 2 v., 1918–19.
4. R. HEGER, *Fünfstellige logarithmische und goniometrische Tafeln* . . . , Leipzig, 1900, p. 83; second edition, 1913, p. 83, $x = [0(1°)179°; 4D]$.

5. J. BAUSCHINGER, *Tafeln zur theoretischen Astronomie*, Leipzig, 1901, p. 142–146. This table does not appear in the 1934 edition. $x = [0(1')3°; 0'' .0001]$, $[3°(1')40°; 0'' .01]$, Δ, as in no. 3.
6. J. R. AIREY, B.A.A.S., *Report 1916*, p. 88–89; $x = [0(.00001).001; 11D]$.
7. HENRI CHRÉTIEN, *Nouvelles Tables des Sinus Naturels spécialement adaptées au calcul des combinaisons optiques* . . . , Paris, 1932, p. 10–27, to $x = .181842$, i.e. to *c.* 10° 25′, 6D critical; then to $x = .52661$, i.e. to *c.* 30° 10′, 5D critical. See *MTAC*, v. 1, p. 16.

Strömgren's two new tables (p. 11–62) are of $x - \sin x$ with arguments x and $\sin x$ each $= [0(.0001).5; 7D]$, Δ. {On p. 63 is a single-page table of $\tan x$ for $x = [0(.001).2; 7D]$, Δ.} The author remarks that Chrétien's tables give sufficient accuracy for most purposes. "However, in investigations of objectives of astronomical instruments of medium, or long, focal length, it is desirable to carry out trigonometrical ray-tracing of higher accuracy. The tables of the present publication have been prepared in order to meet this need." For early arguments, 0(.0001).04, values are given to 8D.

For the argument x in radians the function $x - \sin x$ was tabulated at interval .001, from the 10D table of BAASMTC, *Mathematical Tables*, v. 1, 1931. From this, a corresponding 10D table at interval .0001 was calculated by interpolation. The table thus obtained was checked by differencing. To complete the check a duplicate table was computed by cumulative addition of the differences, and then compared with the original.

Next the table was abridged from 10D to 7D. All tabular values in the 10D table ending in 498, 499, 500, 501, 502 were recomputed to 12D with the aid of the series expression for $\sin x$, and abridged accordingly. Since this revision did not lead to any changes, the 7D values should be correct to the last decimal place.

For values of x up to .04 the table was abridged from 10D to 8D. All tabular values ending in 48, 49, 50, 51, 52 were recomputed to 12D and abridged accordingly.

With $\sin x$ as argument 11D values of $x - \sin x$ were first computed for $\sin x = .01(.01).53$ from the BAASMTC volume referred to above, by a process of backward interpolation utilizing the addition formulae valid for the sine function. These values were checked with the aid of Briggs table of 1633. An interpolation to tenths was carried out, and the values obtained rounded off to 10D, and checked by differencing. Checking and abridgment were then carried out as before.

Pages 4–10 are mostly occupied with an explanation of the use of the tables.

Strömgren has been an ordinary professor of astronomy at the University of Copenhagen and director of the University's astronomical Observatory since 1940.

R. C. A.

321[F].—D. P. BANERJEE, "On a theorem in the theory of partition," Calcutta Math. So., *Bull.*, v. 37, 1945, p. 113–4.

This note contains a short table (for $n \leqq 70$) of the number $Q(n)$ of partitions of n into parts which are both odd and distinct. The author is unaware of a table of WATSON[1] giving this function for $n \leqq 400$. A comparison of the two tables shows no discrepancy. The theorem referred to in the title is to the effect that $p(n)$, the number of unrestricted partitions of n is odd or even according as $Q(n)$ is odd or even. This theorem is noted by Watson[1] and MACMAHON[2] and is actually much older, since it follows at once from the theorem of SYLVESTER[3] that $Q(n)$ is also the number of self-conjugate partitions of n. The recurrence formula used to compute the table is the same as that used by Watson.

D. H. L.

[1] G. N. WATSON, "Two tables of partitions," London Math. So., *Proc.* s. 2, v. 42, 1937, p. 550–556.

[2] P. A. MACMAHON, "The parity of $p(n)$, the number of partitions of n, when $n \leqslant 1000$," London Math. So., *J.*, v. 1, 1926, p. 226.

[3] J. SYLVESTER, "On a new theorem in partitions" and "Note on the graphical method in partitions," Johns Hopkins Univ., *Circulars*, v. 2, 1883, p. 70–71; *Collected Math. Papers*, v. 3, 1909, p. 680–684.

322[G].—Maurice B. Kraitchik, "On certain rational cuboids," *Scripta Mathematica*, v. 11, July–Dec. 1945 [publ. Aug. 1946], p. 317–326. 16.5 × 24.7 cm.

Denoting the edges of a rectangular parallelopiped or cuboid by x, y, z and the diagonals of the faces by X, Y, Z, we shall have

$$x^2 + y^2 = Z^2, \quad y^2 + z^2 = X^2, \quad z^2 + x^2 = Y^2. \tag{1}$$

If m, n, p are integers such that $m^2 + n^2 = p^2$, the following are general solutions:

$$\begin{cases} x = m(p^2 - 4n^2), & y = 4mnp, & z = n(p^2 - 4m^2) \\ X = n(p^2 + 4m^2), & Y = p^3, & Z = m(p^2 + 4n^2). \end{cases} \tag{2}$$

On p. 326 is a table of 50 cuboids which cannot be derived from formulae (2) or from each other. The first and last cuboids of this table are $z = 85 = 5\cdot17$, $x = 4\cdot3\cdot11$, $y = 16\cdot9\cdot5$, $z = 2636361 = 27\cdot7\cdot13\cdot29\cdot37$, $x = 8\cdot5\cdot17\cdot41\cdot107$, $y = 32\cdot3\cdot5\cdot11\cdot17\cdot37$.

Extracts from the article

323[I, O].—I. J. Schoenberg, "Contributions to the problem of approximation of equidistant data by analytic functions. Part A.—On the problem of smoothing or graduation. A first class of analytic approximation formulae", *Quart. Appl. Math.*, v. 4, Apr. 1946, p. 45–99. The tables, p. 91–99, are by Mrs. Mildred Young. 17.7 × 25.4 cm.

The problem of interpolation may be conceived in a very general form as follows: Let $f(x)$ be a function concerning which information is tabulated for a discrete set of values of x, say a sequence $\{x_n\}$. (The function may not even be defined elsewhere.) It is required to define a function $F(x)$ subject to specified restrictions and such that it bears a specified relation to $f(x)$. If the specified relation is that, for every n,

$$F(x_n) = f(x_n), \tag{1}$$

the formula $F(x)$ is called an *ordinary interpolation formula;* on the other hand if we only require (1) to hold approximately, and emphasize the restrictions which make $F(x)$ smooth, $F(x)$ is called a *smoothing* or "modified" *interpolation formula*. There are advantages in understanding the term interpolation broadly enough to take in other forms of relationship, so as to include such matters as mechanical quadratures, but this is irrelevant for our present purposes.

The mth degree Lagrange interpolation formula gives a polynomial $F(x)$ of degree $\leqq m$ which is an ordinary interpolation formula for a set consisting of $m + 1$ distinct points. Under certain assumptions this $F(x)$ is an accurate approximation near the middle of the interval spanned by these $m + 1$ points; this suggests piecing such polynomials together so as to get a formula valid over a wider range. But when this is done the resulting $F(x)$ has discontinuities, either in the function or in its first derivative, at the points where the pieces are joined together. For some purposes this is a grave disadvantage. In dealing with empirical data it is frequently desirable to form a very smooth function $F(x)$ for which (1) holds only to within the accuracy of the data; while $F(x)$ may be computed and smoothed to a far greater accuracy than this, in order to avoid excessive rounding errors in the subsequent calculation. This has led to a more general study of interpolation, to which Schoenberg's paper is a notable contribution.

A general class of interpolation formulae consists of those of the form

$$F(x) = \sum_n f(n)\cdot L_n(x), \tag{2}$$

where the L_n are determined by the x_n independently of $f(x)$. In case the x_n are equidistant and extend to infinity both positively and negatively, it is natural to take $x_n = n$ and

$$L_n(x) = L(x - n), \tag{3}$$

so that (2) becomes

$$F(x) = \sum_{n=-\infty}^{\infty} f(x_n)L(x - n). \tag{4}$$

This is the type considered by Schoenberg. He requires in addition that $L(x)$ be an even function—a restriction which is something of a nuisance since it is not always convenient to maintain it in programming computations for the automatic machines. In Chapters I and II of the present paper he develops a general theory of such formulae using the method of Fourier analysis. In this he makes, for technical reasons, the restriction that $L(x)$ vanish exponentially at infinity. The function $L(x)$ he calls the *base function*. The study of interpolation formulae of type (4) thus reduces to that of their base functions; and it is convenient to speak of such a base function as giving rise to an interpolation formula of such and such type.

Two types of base functions are considered by Schoenberg more in detail. The first type are what the reviewer would call *broken polynomials*, i.e. their graphs are formed by piecing together a finite number of polynomial arcs. This type includes ordinary Lagrangean interpolation when centered and pieced together, as above described, in such a way as to maintain symmetry. Such functions $L(x)$ are characterized by (1) the maximal degree m of the constituent polynomials; (2) the class C^{μ}, i.e. the number μ of continuous derivatives; (3) the highest degree k of polynomials for which $F(x) \equiv f(x)$, and (4) the span s. The second type are analytic functions derived from broken polynomials by considerations in the theory of heat flow. In fact if we set

$$L(x, t) = (\pi t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-(u-x)^2/t} L(u, 0)du, \tag{5}$$

then $L(x, t)$ is a solution of the partial differential equation of heat flow such that

$$\lim_{t\to 0} L(x, t) = L(x, 0). \tag{6}$$

For each fixed t, $L(x, t)$ is then an analytic smoothed form of $L(x, 0)$, the smoothness increasing with t.

In the present Part A (after the general introductory chapters) the broken polynomials are of degree $k - 1$ and class C^{k-2}. Such a curve (polynomial) Schoenberg calls a *spline* curve (polynomial) of order k because the splines used by draftsmen make such curves for $k = 4$. This is the maximum number of continuous derivatives a broken polynomial can have without reducing to an ordinary polynomial. It turns out that a spline of order k must have span at least k. If the span is k the spline is uniquely determined except for a constant factor. With a certain normalization this spline, called a *basic spline of order k*, is $M_k(x)$, viz.

$$M_k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2 \sin \frac{1}{2}u}{u}\right)^k e^{iux}du = \frac{1}{(k-1)!}\,\delta^k x_+^{k-1}, \tag{7}$$

where δ is the central difference operator and

$$x_+^{k-1} = \begin{matrix} x^{k-1} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{matrix} \tag{8}$$

Any spline polynomial can be expressed as a linear combination of basic splines $M_k(x - n)$. The interpolation formula based on $M_k(x)$ is exact for degree 1 and converts every polynomial $f(x)$ of degree $k - 1$ into an $F(x)$ of the same degree.

The analytic base function derived from $M_k(x)$ by (5) is

$$M_k(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_k(u, t)e^{iux}du \tag{9}$$

where

$$\psi_k(u, t) = e^{-\frac{1}{4}tu^2} \left(\frac{2 \sin \frac{1}{2}u}{u}\right)^k. \tag{10}$$

Further functions are then defined thus

$$\phi_k(u, t) = \sum_{n=-\infty}^{\infty} M_k(n, t) \cos nu = \sum_{\nu=-\infty}^{\infty} \psi_k(u + 2\pi\nu, t). \tag{11}$$

$$L_k(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\psi_k(u, t)}{\phi_k(u, t)} e^{iux} du. \tag{12}$$

$$L_k(x, t, \epsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\epsilon + \phi_k(u, t)}{\epsilon + (\phi_k(u, t))^2} \psi_k(u, t) e^{iux} du. \tag{13}$$

Of these $M_k(x, t)$ gives rise to a smoothing interpolation formula which is exact for the degree 1 and preserves the degree $k - 1$; $L_k(x, t)$ gives rise to an ordinary interpolation formula which is exact for the degree $k - 1$; while $L_k(x, t, \epsilon)$, which reduces to $L_k(x, t)$ for $\epsilon = 0$ and to $M_k(x, t)$ as $\epsilon \to \infty$, is a compromise between these two. It increases in smoothing power (in a sense which is explained in the paper) as ϵ increases.

The formula

$$F(x) = \sum_{n=-\infty}^{\infty} f(x_n) L_k(x - n, t, \epsilon) \tag{14}$$

turns out to be equivalent to the following pair (Schoenberg calls the first one f_ν, but we have here used $f(x)$ in another sense)

$$h_\nu = \sum_{n=-\infty}^{\infty} f(n) \omega^{(k)}_{\nu-n}(t, \epsilon) \tag{15}$$

$$F(x) = \sum_{\nu=-\infty}^{\infty} h_\nu M_k(x - \nu, t) \tag{16}$$

where the $\omega_\nu^{(k)}(t, \epsilon)$ are the coefficients in the expansion

$$\frac{\epsilon + \phi_k(u, t)}{\epsilon + (\phi_k(u, t))^2} = \omega_0 + 2 \sum_{\nu=0}^{\infty} \omega_\nu^{(k)}(t, \epsilon) \cos \nu u$$

$$= \sum_{\nu=-\infty}^{\infty} \omega_\nu^{k}(t, \epsilon) e^{i\nu u}.$$

The advantage of this is that $M_k(x, t)$ damps out as $\exp(-x^2)$ whereas $L_k(x, t)$ as $\exp(-x)$ only.

The tables in the Appendix give certain values for $k = 4$, $t = .5$. These tables, computed by Mrs. Young, are as follows:

Table I: $M_4(x, \frac{1}{2})$, $M_4'(x, \frac{1}{2})$, $M_4''(x, \frac{1}{2})$, for $x = -5(.1) + 4$. These are conveniently arranged for subtabulating the given table $f(n)$ to tenths; values where the arguments differ by an integer are in the same column.

Table II: $\omega_n^{(4)}(\frac{1}{2}, \epsilon)$ for $\epsilon = 0(.1)1$, $n = [0(1)26; 8D]$. (Note that $\omega_n^{(k)}$ is symmetric in n.)

Table III: $L_4(x, \frac{1}{2}, \epsilon)$ and $L_4''(x, \frac{1}{2}, \epsilon)$ for $\epsilon = 0(.1)1$, $x = [0(.5)26; 8D]$. This table is for use with (14) for subtabulation to halves. On account of the slow damping of $L_k(x, t, \epsilon)$ use of (15) and (16) with Tables I and II is recommended for subtabulation to tenths.

HASKELL B. CURRY

Pennsylvania State Coll.

324[K].—SIEGFRIED KOLLER, *Graphische Tafeln zur Beurteilung statistischer Zahlen*. Dresden and Leipzig, Steinkopff, second ed. 1943. Lithoprinted by Edwards Bros., Ann Arbor, Michigan, 1945, x, 73 p. 19.5 × 27.4 cm. $3.00. Published and distributed in the public interest by authority of the Alien Property Custodian under license no. A-634.

This book presents in graphical form the numerical information required for some of the most usual routine tests of statistical significance. Except for insignificant variations

the book covers only material generally used by statisticians but usually presented by means of numerical tables. In the preface to the second edition the author complains that the graphical method usually meets a certain resistance; it must be said, nevertheless, that in the present case its advantages are far from obvious. According to the author the graphs require "considerably less" space than numerical tables. The justification of this claim could be disputed on the basis of a comparison of some graphs in the book under review with the corresponding tables in R. A. FISHER, *Statistical Methods for Research Workers* (eighth ed., Edinburgh and London, 1941), which is generally used by statisticians. The principal advantage of the graphical method, it is said, is that it does not require numerical interpolation. This claim is very general. In the particular case of the statistical tables with which we are concerned the accuracy required is so small and the spacing of usual tables is so dense that, in the reviewer's opinion, numerical interpolation does not present a problem to speak of. Instead of interpolation the present graphs require one to find and follow a curve in a very densely drawn family of curves and to read off one of its ordinates by means of a dense and rather unsharp coordinate net on a non-uniform scale; alternatively to join two points by a straight line and to read off the position of its intersection with a third line provided with a scale. For the last purpose the author suggests the use of a glass ruler on which a line is marked. The use of the usual transparent rulers or triangles is said to lead to inaccuracies (where they are used, a second reading with the upper side turned down is said to be necessary). A few trials made the reviewer doubtful as to the claim that this procedure is "more handy, more convenient, and faster than the use of numerical tables."

The book contains an introduction (p. 1–13) explaining the fundamental notions. There are in all seventeen graphs, all of which appear on odd numbered pages, mostly with the reverse blank. Facing each graph are directions for use and empirical examples. Wherever required, the odd page following the graph contains a mathematical definition of the function presented. In the statistical graphs the arbitrary "confidence level" has been chosen as $\epsilon = .0027$. This value corresponds to the old fashioned 3σ-rule and is used instead of the one and five per cent levels customary in the American and British literature. Special attention is paid to small samples; large number approximations are used only when they prove to be within the error limits of the graphs. The question of accuracy is not discussed in detail.

Only **Table 7** presents a simple graph of a function: the ϵ-point of Student's t-distribution as a function of the degrees of freedom n for $n > 10$. For $n \leqslant 10$ the values are given numerically to 2D; almost the same accuracy can be obtained from the graph. The remaining graphs are of three different types.

Group 1. Here functions of one variable are represented by means of two scales (divisions) along the same straight line. The scales are non-uniform since the spacing of the lines of division has had to be adjusted to the requirements of readability. **Table 2** gives x^2 for $1 < x < 10$ along five segments of some 19 cm. each. **Table 9** gives the ϵ-point of the χ^2-distribution with a number of degrees of freedom $n > 40$ (20 cm.). For $n \leqslant 40$ the values are given numerically in a short table which makes the accompanying graph look almost ridiculous. The same arrangement is found in **Table 10** giving $r = t(t^2 + n)^{-\frac{1}{2}}$ where t is the function of n mentioned in the description of **Table 7**. Values for $n \leqslant 30$ are given numerically, for $n > 30$ along a 19 cm. long scale. **Table 11a** represents (along some 90 cm.) the function

$$z = \tfrac{1}{2} \ln \frac{1 + r}{1 - r}$$

which is of use in correlation theory. Similarly, **Table 14** gives the normal density function for $0 < t < 4$, and **Table 15** its integral.

Group 2 contains nomograms of functions $z = f(x, y)$ of two variables. Here each of the variables is represented by the subdivision of one of three parallel lines. The value of z is found by bringing the z-line to intersection with the line joining the given x- and y-points. Thus **Tables 1a** and **1b** serve to find the product or quotient of two numbers.

Table 6 lacks mathematical significance; it is an auxiliary to **Table 5** and permits to reduce samples of unequal size to equivalent samples of equal size. **Table 8** serves to compute $z = (x^2 + y^2)^{\frac{1}{2}}$. **Table 11b** represents

$$3\left(\frac{1}{x-3}+\frac{1}{y-3}\right)^{\frac{1}{2}}.$$

The square root is the standard deviation of the difference of the z-values belonging to the correlation coefficients for two samples of sizes x and y. The z-values are obtained from **Table 11a**. These tables serve to test the significance of the difference between two correlation coefficients. To the second group belongs also **Table 12**, serving to compute partial correlations, and giving the function of three variables $r = (\alpha - \beta\gamma)[(1 - \beta^2)(1 - \gamma^2)]^{-\frac{1}{2}}$. Here the α-point and the r-point are found as described above for x and z while instead of the simple y-point one has now to find the intersection of a β-curve and a γ-curve.

Group 3. This group too is concerned with functions $y = f(x, n)$ of two variables, but this time they are represented by the one-parametric families of curves obtained by keeping n fixed. **Table 3** serves to test the hypothesis that a given sample comes from a Bernoulli population with probability p. For various n the curves represent $p_u - p$ as function of p, where np_u is the least number such that the probability of np_u or more successes is not greater than $\epsilon/2$. The hypothesis is to be rejected if the observed number of successes lies outside the interval (np_u, np_l), where p_l is the point on the p_u-curve with abscissa $1 - p$. **Table 4** gives the ϵ-confidence limits for estimating the unknown parameter p from the number of successes in a sample of n (cf. C. J. CLOPPER & E. S. PEARSON, "The use of confidence or fiducial limits illustrated in the case of the binomial," *Biometrika*, v. 26, 1934, p. 404f). **Table 5** serves to test the significance of the difference of the number of successes in two samples of n trials each. The curves represent the maximum permissible difference under the arbitrary assumption that the true value of p is exactly the arithmetic mean of the two frequencies of successes. Such an assumption is not justified by theory. However, even if it is accepted the test will, in general, lead to two different results depending on whether one starts with the smaller observed frequency and estimates an upper bound for the larger one, or starts with the larger frequency and estimates the smaller one. Finally, **Table 13** illustrates Fisher's z-distribution on which analysis of variance is based.

W. FELLER

Cornell University

325[L].—GERTRUDE BLANCH, "On the computation of Mathieu functions," *J. Math. Phys.*, v. 25, 1946, p. 1–20. 17.2 × 25.3 cm.

Techniques for computing the characteristic numbers and Fourier coefficients of the periodic Mathieu functions are now well known. The present paper gives a method by which the accuracy of the characteristic number may be rapidly and *systematically* improved.

To fix ideas, consider an even solution of

$$d^2y/dt^2 + (\alpha - 2\theta\cos 2t)y = 0 \tag{1}$$

of period π, which may be expressed in the form

$$y = \sum_{r=0}^{\infty} B_{2r}\cos 2rt. \tag{2}$$

In order that (1) may be satisfied, the B_m must satisfy a three-term recurrence relation, and writing

$$G_m = B_m/B_{m-2},\quad H_m = 1/G_m,\quad k_m = (\alpha - m^2)/\theta \tag{3}$$

this may be written in the alternative forms,

$$G_m = k_{m-2} - H_{m-2} \text{ for } m \geq 5 \tag{4a}$$

$$G_m = 1/(k_m - G_{m+2}) \text{ for } m \geq 3 \tag{4b}$$

together with

$$G_2 = k_0, \quad G_4 = k_2 - 2H_2 = 1/(k_4 - G_6). \tag{4c}$$

To satisfy these, α must have one of a set of characteristic values. If α' is an approximation to one of these, the relations (4c) and (4a) apparently determine in succession $G_2, G_4, \cdots$. But this is not, in fact, the case, for if (2) is to converge, then ultimately $|G_m| < 1$, and since k_m increases rapidly, so also does H_m, with the result that significant figures are rapidly lost. Hence we must also start from some large value of m (so large that B_m is insignificant) and come backwards by use of (4b), meeting at H_p, where p is some convenient value of m (corresponding generally to the coefficient B_p greatest in absolute value). Agreement between the values of H_p determined forwards and backwards is a test of the accuracy of α', but the most significant contribution in this paper is the masterly way in which the discrepancy between these values is made to yield a better value of α, by an application of what is essentially Newton's method of successive approximation to the root of an equation.

If $\alpha = \alpha' + \lambda$ is the true value, and H_{p1} and H_{p2} are the values of H_p determined respectively by using (4a) forwards and (4b) backwards, then it is shown that, if squares and higher powers of λ are negligible,

$$\lambda = \theta(H_{p1} - H_{p2})/(R_{p1} + R_{p2}) \tag{5}$$

where

$$R_{p1} = (B_{p-2}^2 + B_{p-4}^2 + \cdots + 2B_0^2)/B_p^2 \tag{6a}$$

$$R_{p2} = (B_p^2 + B_{p+2}^2 + \cdots)/B_p^2 \tag{6b}$$

(with slight modifications if $p = 0$ or 2), and the consequent changes in the G and the H are listed.

Formulae for the other three classes of function differ only in detail.

Complete numerical details are given for two examples. The first exhibits the method, and the enormous increase in accuracy which can accrue if α' is really close to α, 9D accuracy being converted into at least 18D accuracy by only one application—by a complete recalculation of the G and H, using the new value of α. The second shows how to use the formulae for the errors in the H if only a slight (9D converted to 12D) increase in accuracy is desired. The second example also shows some points which arise when (as for higher orders) the G_m do not decrease uniformly in numerical value with increasing m.

W. G. Bickley

326[L].—N. J. Durant, "Struts of variable flexural rigidity," *Phil. Mag.*, s. 7, v. 36, Aug. 1945, p. 572–577 [publ. Apr. 1946]. 17 × 25.3 cm.

The differential equation

$$EId^2y/dx^2 + Py = 0$$

for a strut of variable cross-section (E is Young's modulus, I the second moment of the cross-section, y the transverse deflection at a distance x measured along the strut, and P the longitudinal thrust) can be integrated in terms of Bessel functions if $I = I_0e^{-kx/l}$ (I_0, k, constants, l the length of the strut); this is a reasonable approximation in the case of some engineering structures. If the strut is fixed at one end and free at the other, the critical thrust is $\gamma EI_0/l^2$, where $\gamma = (k\theta/2)^2$, θ is the smallest positive root of the equation

$$J_0(\beta\theta)Y_1(\theta) - Y_0(\beta\theta)J_1(\theta) = 0$$

in which $\beta = e^{\frac{1}{2}k}$. Table I, p. 576, gives k to 5S; and β, θ, $\beta\theta$, γ and 4γ to 4D, for $c = e^{-k} = .025, .05, .1(.1)1$.

W. G. Bickley

Imperial College
London, S. W. 7

327[L].—H. B. Dwight, "Table of the Bessel functions and derivatives J_2, J_1', J_2', N_2, N_1', N_2'," *J. Math. Phys.*, v. 25, May, 1946, p. 93–95. 17.2 × 25.3 cm.

Here are tables of $J_2(x)$, $J_1'(x)$, $J_2'(x)$, $Y_2(x)$, $Y_1'(x)$, $Y_2'(x)$ [we see no reason for using German notation], for $x = .01(.01).2(.1)10$. J_2' for $x = 2(.1)10$, and J_2 and J_1' throughout, are to 8D; J_2', for $x = .01(.01).2(.1)1.9$, are to 6 or 7D. Y_1' and Y_2', for x from 1.4 to 10, are to 6D; and for other values of x, to 6 or 7S. Y_2 are for $x < .2$ to 6S; for $.2 \leq x < 5$, to 7S; for $x \geq 5$ to 6D.

328[L].—V. Fok, "The distribution of currents induced by a plane wave on the surface of a conductor," Akad. Nauk USSR, Moscow, *J. of Physics*, v. 10, no. 2, 1946, p. 135–136.

The tables are identical with those already described in RMT **309**.

329[L].—F. I. Frankl, "K teorii sopel Lavali͡a" [On the theory of the Laval nozzle], Akad. Nauk USSR, Moscow, *Izvestii͡a, serii͡a matem.*, v. 9, p. 421, Nov. 1945.

There are here three tables to 4D.

I. $Z_1(t) = F(\frac{1}{3}, -\frac{1}{3}; \frac{1}{2}; t)$

$$= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})} \int_0^1 u^{-\frac{2}{3}}(1-u)^{-\frac{5}{6}}(1-ut)^{\frac{1}{3}}du$$

for $t = -.5(.1)+1$.

II. $Z_2(t) = -F(\frac{1}{3}, -\frac{1}{3}; \frac{1}{2}; 1-t)$

for $t = 0(.1)1.5$.

III. $Z_3(t) = F(\frac{1}{3}, -\frac{1}{3}; \frac{1}{2}; 1-t) - F(\frac{1}{3}, -\frac{1}{3}; \frac{1}{2}; t)$

for $t = 0(.1)1$.

330[L].—E. Niskanen, "On the deformation of the earth's crust under the weight of a glacial ice-load and related phenomena," Suomalainen Tiedeakatemia, Helsingfors, *Toimituksia*, *Annales*, s. A., III. *Geologica, Geographica*, no. 7, 1943.

Table II, p. 34–36, gives the values of $P_n(x)$, and $T_n(x) = P_{n-1}(x) - P_{n+1}(x)$, for $n = [0(1)61; 6D]$, $x = \cos\theta$, $\theta = 6°, 9°, 30°$. The values of P_n, $n = 1(1)32$, were taken from H. Tallqvist, *Sechsstellige Tafeln der 32 ersten Kugelfunktionen* $P_n(\cos\theta)$, Soc. Scient. Fennicae, *Acta*, n.s. A, v. 2, no. 11, 1938, p. 3–43, and the values of $\cos\theta$ were taken from J. T. Peters, *Einundzwanzigstellige Werte der Funktionen Sinus und Cosinus zur genauen Berechnung* . . . , Akad. d. Wissen., Berlin, *Abh.*, *Phys.-Math. Kl.*, 1911, Anhang, no. 1. The terms 33(1)61 were computed by means of the formula

$$P_n = [(2n-1)/n]P_{n-1} - [(n-1)/n]P_{n-2}.$$

For $n = 52(1)61$, some final figures in parentheses "can be erroneous," so that, e.g., for $n = 61$ the last three or four places of values of the functions are of doubtful accuracy.

331[L].—Jørgen Rybner, "Fourieranalyse af frekvensmodulerede Svingninger med Savtakvariation af Øjebliksfrekvensen (Kipmodulation)," *Matematisk Tidsskrift B*, 1946, *Festskrift til N. E. Nørlund*, second part, p. 97–112. 15.2 × 24.1 cm.

Let Ω, $\Delta\Omega$, f, be positive numbers, and let $\Phi(t)$ be a function of period $1/f$ which in the interval $-\frac{1}{2}f \leqslant t \leqslant \frac{1}{2}f$ equals $f\Delta\Omega t^2$. By using the developments

$$\cos f\Delta\Omega t^2 = \tfrac{1}{2}a_0 + a_1\cos\omega t + a_2\cos 2\omega t + \cdots,$$
$$\sin f\Delta\Omega t^2 = \tfrac{1}{2}a_0' + a_1'\cos\omega t + a_2'\cos 2\omega t + \cdots,$$

where ω is defined by the relation $2\pi/\omega = 1/f$, the author represents the function $u = \sin [\Omega t + \Phi(t)]$, "Kipmodulation," in the form

$$u = \sin \Omega t(\tfrac{1}{2}a_0 + a_1 \cos \omega t + \cdots) + \cos \Omega t(\tfrac{1}{2}a_0' + a_1' \cos \omega t + \cdots).$$

The coefficients a_n, a_n' here depend on the parameter m (modulation index) defined by the formula $m = \Delta\Omega/\omega$. Using Lommel's formulae for Bessel functions the author tabulates (T. 1) the numbers $\tfrac{1}{2}a_n(m)$, $\tfrac{1}{2}a_n'(m)$, $M = \tfrac{1}{2}(a_n^2 + a_n'^2)^{\frac{1}{2}}$, M^2, and $y = m\pi = [3(3)30; 6D]$, $n = 0(1)4$. Table 2 gives the values for $m = 5$, $n = [0(1)10; 6D]$.

A. ZYGMUND

Univ. of Pennsylvania

332[L].—OLOF E. H. RYDBECK, "On the propagation of radio waves," Chalmers Tekniska Högskola, *Handlingar*, no. 34, 1944. 170 p. 17.5 × 24.8 cm.

On p. 160–166 are "Tables of cylinder functions of order $\pm \tfrac{1}{3}$ and $\pm \tfrac{2}{3}$." There are tables of the following 12 functions for $x = [0(.02)1(.2)8; 4\text{–}5S]$: $J_{\pm 2/3}(x)$, $Y_{2/3}(x)$, $I_{\pm 2/3}(x)$, $I_{\pm 1/3}(x)$, $|H_{2/3}^{(1)}(x)|$, $Y_{2/3}(xe^{-\frac{1}{2}3i\pi})$, $|H_{2/3}^{(1)}(xe^{-\frac{1}{2}3i\pi})|$, $iY_{1/3}(xe^{-\frac{1}{2}3i\pi})$, $|H_{1/3}^{(1)}(xe^{-\frac{1}{2}3i\pi})|$. For the same range of x there are also tables of Phase $H_{2/3}^{(1)}(x)$, Phase $H_{2/3}^{(1)}(xe^{-\frac{1}{2}3i\pi})$, and Phase $H_{1/3}^{(1)}(xe^{-\frac{1}{2}3i\pi})$, each to the nearest 1′.

Rydbeck refers to G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 1922, where there are tables of $J_{\frac{1}{3}}(x)$, $Y_{\frac{1}{3}}(x)$, $|H_{\frac{1}{3}}^{(1)}(x)|$, which are tabulated for $x = [0(.02)16; 7D]$. He also mentions tables of $J_n(x)$, $I_n(x)$, $n = \pm \tfrac{1}{3}, \pm \tfrac{2}{3}$, for $x = [0(.2)8; 4D]$, by A. DINNIK, in *Archiv Math. Phys.*, s. 3, v. 18, 1911, p. 337–338. No such tables are given at this reference, but only tables of $n!J_n(x)$, $n = \pm \tfrac{1}{3}, \pm \tfrac{2}{3}$. For information concerning Dinnik tables of $J_n(x)$, $I_n(x)$, $n = \pm \tfrac{1}{3}, \pm \tfrac{2}{3}$, in 1914 and 1933, see *MTAC*, v. 1, p. 286, 287; for other tables see also p. 237–238.

333[L, M].—GREAT BRITAIN Nautical Almanac Office, *Tables of the Incomplete Airy Integral*, for the Department of Scientific Research and Experiment, Admiralty Computing Service, April 1946. ii, 15 p. + 1 folding plate. 20.3 × 30.5 cm. Reproduced by photo offset from typescript. This publication is available only to certain Government agencies and activities. Erratum slip for p. 7, dated May 1946.

This report contains values (p. 11–15) of the incomplete Airy integral

$$F(x, y) = \pi^{-1}\int_0^\pi \cos (xt - yt^3)dt \tag{1}$$

for $x = [-2.5(.1) + 4.5; 4D]$, $y = 0(.02)1$. The last figure tabulated should not be in error by more than one unit; except for small y and negative x the error is unlikely to exceed .6.

The tables can be interpolated in the x-direction using second differences, but for accurate interpolation in the y-direction fourth differences are required; if second differences only are used, the maximum error to be expected is two units.

If new variables $Y = \sqrt[3]{3y}$ and $X = x/Y$ are introduced then, with $T = Yt$, the integral for $F(x, y)$ may be rewritten in the form

$$F(x, y) = (\pi Y)^{-1}\int_0^{\pi Y} \cos (XT - \tfrac{1}{3}T^3)dT \tag{2}$$

which compares immediately with

$$Y^{-1}Ai(-X) = (\pi Y)^{-1}\int_0^\infty \cos (XT - \tfrac{1}{3}T^3)dT. \tag{3}$$

Here the complete Airy integral Ai is one of the solutions of the differential equation

$$\frac{d^2v}{dx^2} = xv. \tag{4}$$

$Ai(x)$, with an appropriately defined second solution $Bi(x)$, has been tabulated by J. C. P. Miller, see *MTAC*, v. 1, p. 283.

The corresponding differential equation satisfied by the incomplete Airy integral is easily shown to be

$$\frac{\partial^2 F}{\partial x^2} + \frac{x}{3y} F = \frac{1}{3\pi y} \sin (x\pi - y\pi^3). \tag{5}$$

3. *Method of computation*, p. 2–4; 4. *Numerical integration of the differential equation*, p. 4–9. The opportunity is taken here to describe in some detail a method, used in H. M. Nautical Almanac Office for some time, for the numerical integration of second order linear differential equations. The principle seems to be largely due to B. V. NUMEROV ("Méthode nouvelle de la détermination des éphémérides en tenant compte des perturbations," Observatoire Astrophysique Central de Russie, *Publications*, v. 2, Moscow, 1923), who published many papers from 1923 onwards developing the method and applying it to the simultaneous integration of the three (non-linear) second-order equations defining the motion of a particle under the Newtonian attraction of the bodies in the solar system.

Extracts from introductory text

334[L, M].—GREAT BRITAIN, Nautical Almanac Office, Department of Scientific Research and Development, Admiralty Computing Service, *Tabulation of the Function* $f(x, y) = \int_0^\infty \frac{e^{-k}\{J_0(kx)\cosh(ky) - 1\}}{\sinh k}\,dk$. Offprint reproduction of handwriting and typescript tables; 7 p. + 1 folding plate. No. SRE/ACS 47, February, 1945. 20 × 33 cm. These tables are available only to certain Government agencies and activities.

The function $f(x, y)$ is the solution to a two-dimensional potential problem, being that solution in the strip $0 \leqslant y \leqslant 1$ of the differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial y^2} = 0, \tag{1}$$

which satisfies the conditions

$$\frac{\partial f}{\partial y} = 0 \text{ on } y = 0 \text{ for } x \neq 0, \ \frac{\partial f}{\partial y} = 1/(x^2 + 1)^{3/2} \text{ on } y = 1,$$

and $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sim O[1/(x^2 + y^2)]$ near the origin and also for large values of x.

$f(x, y)$ is tabulated to 4D in the ranges

$$x = 0(.1)5, \quad y = 0(.1)1, \quad \Delta_x'', \quad \Delta_y''.$$

The methods of computation employed, and the checks which were applied, suggest that the last figure given is seldom in error by more than one unit, and never by more than two units.

The integral was evaluated in two stages. First, values were calculated from the series

$$f(x, y) = \sum_{n=1}^{\infty} \left\{ \frac{1}{[x^2 + (2n - y)^2]^{\frac{1}{2}}} + \frac{1}{[x^2 + (2n + y)^2]^{\frac{1}{2}}} - \frac{1}{n} \right\} \tag{2}$$

at interval .2, on the bounding lines $x = 0$, $x = 5$, $y = 0$, $y = 1$. These values were then used as boundary conditions for the differential equation (1), and the function obtained in the interior of the rectangle by relaxation methods.

The values at interval .2 were interpolated to an interval of .1, an interval more convenient for tabulation and subsequent interpolation. For interpolation purposes, the second differences in both directions are tabulated on the same line as the function.

On the folding plate are "Values and contours of $f(x, y)$," unit 10^{-4}.

Extracts from introductory text

335[L, M].—HARVARD UNIVERSITY, Computation Laboratory, *Annals*, v. **2**: *Tables of the Modified Hankel Functions of Order one-third and of their Derivatives*. Cambridge, Mass., 1945. xxxvi, 235 p. 20.2 × 26.5 cm. $10.00. See MAC **24**.

Bessel functions of orders $\frac{1}{3}$ and $-\frac{1}{3}$ for real and pure imaginary arguments have been very extensively tabulated by the N.Y.M.T.P. For the case of complex arguments very little has been done. (See BATEMAN & ARCHIBALD—*Guide to Tables of Bessel Functions*, *MTAC*, v. 1, p. 205–308, 1944). The volume under review goes a long way towards filling the existing lacuna.

The differential equation

$$\frac{d^2u}{dz^2} + zu = 0$$

possesses a general solution of the form

$$u = Az^{\frac{1}{2}}J_{\frac{1}{3}}(\tfrac{2}{3}z^{3/2}) + Bz^{\frac{1}{2}}J_{-\frac{1}{3}}(\tfrac{2}{3}z^{3/2}).$$

As the authors remark, the direct tabulation of the Bessel Functions of orders $\pm\frac{1}{3}$ is inadvisable since these functions are not single-valued. For this reason and having in mind the solution of a certain problem in wave propagation, the authors chose for tabulation the functions

$$h_1(z) = \frac{k}{\pi i}\int_{-\infty}^{\infty e^{\frac{1}{3}i\pi}} e^{zt+\frac{1}{3}t^3}dt = g + \tfrac{1}{3}i\sqrt{3}(g - 2f)$$

$$h_2(z) = \frac{k^*}{-\pi i}\int_{-\infty}^{\infty e^{-\frac{1}{3}i\pi}} e^{zt+\frac{1}{3}t^3}dt = g - \tfrac{1}{3}i\sqrt{3}(g - 2f)$$

where k and k^* are constants and

$$f = (\tfrac{2}{3})^{\frac{1}{3}}z^{\frac{1}{2}}J_{-\frac{1}{3}}(\tfrac{2}{3}z^{\frac{3}{2}})$$
$$g = (\tfrac{2}{3})^{\frac{1}{3}}z^{\frac{1}{2}}J_{\frac{1}{3}}(\tfrac{2}{3}z^{\frac{3}{2}}).$$

It may be remarked that the functions $h_1(z)$ and $h_2(z)$ above defined are linear combinations of

$$\Lambda_{-\frac{1}{3}}(\tfrac{2}{3}z^{\frac{3}{2}}) \quad \text{and} \quad z\Lambda_{\frac{1}{3}}(\tfrac{2}{3}z^{\frac{3}{2}})$$

where

$$\Lambda_\nu(z) = \Gamma(\nu + 1)J_\nu(z)/(\tfrac{1}{2}z)^\nu.$$

Moreover it may be shown that $\Lambda_\nu(\frac{2}{3}z^{\frac{3}{2}})$ where $\nu = \pm\frac{1}{3}$ is a well behaved function of z in the neighborhood of the origin, and would have been just as suitable for tabulation as the functions adopted by the authors.

The volume contains six tables, four short ones and two long ones. The contents follow:

Table I: First 23 coefficients (11 or 12D) of the Maclaurin series for f, g, f', g'.
Table II: First 14 coefficients in the asymptotic series for $h_1(z)$ and $h_2(z)$.
Table III: Various constants.
Table IV: Zeros of $h_2(z)$ and $h_2'(z)$; $|z_0| < 6$.
Table V: $h_1(z)$ and $h_1'(z)$ to 8D; $\Delta x = \Delta y = 0.1$; $|x + iy| \leqq 6$.
Table VI: $h_2(z)$ and $h_2'(z)$ to 8D; $\Delta x = \Delta y = 0.1$; $|x + iy| \leqq 6$.

In tables V and VI the values of the functions are given only for the upper half of the z-plane; values in the lower half may be obtained from the given entries by means of simple "reflexion" formulae given with the tables. Beyond $|x + iy| = 6$ the asymptotic expan-

sions included in the introduction furnish convenient means of computing $h_1(z)$ and $h_2(z)$, though not always, with an accuracy comparable to the tabular values. The volume contains an informative introduction describing various properties of the functions and their relations to other functions such as Airy integrals and Bessel Functions, of various fractional orders. The introduction also contains contour lines for $R(h_1)$ = const. and $I(h_1)$ = const. as well as a number of other useful graphs.

Even though the interval of tabulation of $h_1(z)$ and $h_2(z)$ is fairly coarse, the problem of interpolating to the full accuracy of the table is possible though somewhat difficult because of the complicated interpolation formulae needed. By utilizing the properties of harmonic functions the interpolation formulae may be considerably simplified for purposes of subtabulation.

Each value in the table was computed ab initio from the power series expansions and checked by duplicate calculation utilizing different equipment wherever possible. A partial check was afforded by the "Wronskian" relation.

A useful bibliography on Bessel Functions of order $\frac{1}{3}$ and Allied Functions is included. This volume is the first in a series of volumes on Bessel Functions computed on the "Automatic Sequence Controlled Calculator." The result is certainly impressive when one considers that according to the authors it took the machine merely the equivalent of 45 days to complete the tables. It may well be said that this basic table is the first in the new era of high speed computing techniques.

William Horenstein

NYMTP

Editorial Note: The next six volumes of the Harvard *Annals of the Computation Laboratory*, for which computations have been completed, are to be as follows: **3.** $J_0(x)$, $J_1(x)$, and **4.** $J_2(x)$, $J_3(x)$, for x = [0(.001)25(.01)100; 18D], rounded off from computations to 23D. **5.** $J_4(x)$, $J_5(x)$, $J_6(x)$; **6.** $J_7(x)$, $J_8(x)$, $J_9(x)$; **7.** $J_{10}(x)$, $J_{11}(x)$, $J_{12}(x)$; **8.** $J_{13}(x)$, $J_{14}(x)$, $J_{15}(x)$, all for x = [0(.001)25(.01)100; 10D]. These six volumes are each to contain about 600 p. For a similar range it is planned to issue further volumes tabulating $J_n(x)$, n = 16(1)100. The director of the Computation Laboratory and the editor of the *Annals* is Howard H. Aiken, associate professor of applied mathematics in Harvard University. The staff is housed in a new two and one-half story building, specially built for its activities. There will be ample room for other machines to be added to their remarkable Automatic Sequence Controlled Calculator (see *MTAC*, v. 2, p. 91, and *MAC* **24**) set up in a square room about 60 feet on a side.

336[V].—NYMTP, *Table of* $F(v) = \dfrac{1}{v}\dfrac{d}{dv}\ln J_6(v)$. No. OP265–2–97–11. New York, 1942. 11 leaves on one side of each leaf. Mimeographed. 20.7 × 33 cm. This publication is available only to certain Government agencies and activities.

The values of $F(v)$ are tabulated as a function of u where $u = v^2/100$ (v in meters per second), and where J_6 is the Army retardation function, determined experimentally, for the 3-inch Common Steel Shell Model 1915 as tabulated by the Aberdeen Proving Ground, Maryland, in 1932.

The NYMTP table of $F(v)$ differs from the 1932 Army table of $F(v)$, no. N-1-35, tabulated at the Army Proving Ground, Aberdeen, Maryland, in intervals as follows:

Army table: u = 0(10)800(1)1500(10)4000(100)8600.
NYMTP table: u = 0(1)100(2)500(10)1000(2)2000(10)8600;
6D to 89; 7D to 1410; 8D to 3830; 9D to 8600.

The function $F(v)$ occurs in various formulae for the computations of differential variations in velocity, air density, wind weighting factors, ballistic coefficient, etc., which are applicable in modifying computed trajectories of projectiles. If x and y represent the coordinates of a projectile on the original computed trajectory, x and y satisfy the differential equations $x'' = -Ex'$ and $y'' = -Ey' - g$, where $E = J_6(v)H(y)/C$; C is the ballistic coefficient. The coordinates on the modified trajectory at the instant t are $x + \xi$ and $y + \eta$.

A formula for δE, the part of the change in the retardation due to the variations in x', y' and y, is[1]

$$\delta E = E[F(v)\cdot v\Delta v + \eta\cdot d \ln H/dy]$$

where $v\Delta v = (x'\xi' + y'\eta')$, $H(y) = e^{-.0001036y}$. Since numerical integration must be used in determining the differential corrections to be applied at each point of the trajectory it is necessary to have tabulated values of $F(v)$.

PAUL D. THOMAS

Bureau of Ordnance, Navy Department

[1] D. JACKSON, *The Method of Numerical Integration in Exterior Ballistics*, Washington, 1921, p. 24; this was a text-book prepared in the office of the Chief of Ordnance, 1919.

EDITORIAL NOTE: These tables were later superseded by smoother functions based on more recent firings. In Jackson's publication is an extended table of $H(y)$.

MATHEMATICAL TABLES—ERRATA

References have been made to Errata in RMT **319** (Duffield, Lefort, Vega), **332** (Rydbeck); N**62** (Corey, Hardy & Rogosinski, Harvard, Zygmund, etc.).

88. H. T. DAVIS, *Tables of the Higher Mathematical Functions*, v. 2, Bloomington, Indiana, 1935, p. 29.

There are three serious errors on this page, in $\psi'(x)$ which Davis defines as $d^2 \ln \Gamma(x)/dx^2 = d^2 \ln (x-1)!/dx^2$, and in $\log \psi'(x)$.

For $\psi'(.05) = 401.552357\ 342115$, *read* $401.532357\ 342115$;
For $\psi'(.11) = 84.077927\ 249967$, *read* $84.059535\ 747392$;
For $\log \psi'(.11) = 1.92468\ 19966$, *read* $1.92458\ 69871$.

J. W. WRENCH, JR.

4211 Second St. N.W.
Washington 11, D. C.

89. FMR, *Index*, 1946. See *MTAC*, v. 2, p. 13–18, 136.

A. Apart from a few definite errata noted below we have also indicated inconsistencies, indefinite statements, and a couple of notation changes, which the authors may desire to consider when preparing a new edition. The authors clearly recognized that slight blemishes of this kind existed, because of various elements entering into the preparation and publication of their work. Changes are in italics. See also MTE 90.

P. 23, 2.3 Higher *Positive* Integral Powers [2.5 Higher Negative . . .].
P. 25, **2.3** Higher *Positive* Integral.
P. 33, l. 7, $n = 440\times 2^p$.
P. 35, **3.14** 27 dec. Thoman.
P. 48, **4.18**, l. 4, 4.02*1*.
P. 51, **4.41**, *for* 4.412, *read* 4.4121; *for* 4.413, *read* 4.4132.
P. 76, **4.9333**, *for* 2^m, *read* 2^q.
P. 100, last l., Cauchy 1882, why not 1827? Also p. 101, **5.7115**.
P. 111, last l., *de* Decker; p. 124 and 125, 5 d., *de* Lella.
P. 151, **9.24**, l. 2, δ for d (longitude)? [see **9.23**].
P. 192, **13.4** 4 dec. *for* 10^4, 10^5, 10^6, *read* $10^{4(1)6}$?
P. 200, 7 dec. Brownlee 1923 (Russell, which one?).
P. 208, **14.92**, *for the heading* "Tables of x," *read* "Inverse tables relating to $B_x(p, q)$"?
P. 251, **17.33** heading to make uniform with **17.35**, *read* $G_0(x)$ and $G_1(x)$, then on next line $G_n(x) = -\frac{1}{2}\pi Y_n(x)$.

P. 252, **17.341** $G_n(x)$: General Tables [to make uniform with **17.361**].
P. 262, **17.7212** *read* $\log_{10}|J_1(j_{0,s})|$; also, p. 263, **17.7312**, *read* $\log_{10}|J_0(j_{1,s})|$.
P. 265, **17.751**, *for* n Integral, *read* m Integral.
P. 278, before **18.61** enter **18.6**. Expressions involving both $J_n(x)$ and $I_n(x)$.
P. 326, **22.0**. Introduction, as heading and subheading, omitted.
P. 368, after l. 5, **24.0** Introduction omitted; in **24.21**, l. 1, *for* requisite, *read* requisites.
P. 412, after Lohmann insert reference to T. Lohnstein 1892 [see p. 429, Runge 1891].
P. 435, Steinhauser 1865, l. 2, fünfzehnstellige*r*.
P. 437, Terrill & Sweeny, no cross-reference from Sweeny.

There are many places in Part I where initials seem necessary, exactly to identify individuals in Part II. Such, for example, are the following: *K.* Pearson (p. 21, 24), *C. F.* Gauss (p. 22), *P. L. H.* Davis (p. 144), *W. P.* Elderton (p. 200), *J. B.* Russell (p. 228), *G. W.* Hill and *G. N.* Watson (p. 316), *J. G.* Schmidt (p. 317), *E. A.* Milne and *J. C. P.* Miller (p. 332), *G. N.* Watson (p. 343), *C. J.* Hill (p. 344), *Ch.* Jordan ? (p. 352), *K.* Pearson (p. 355), *R. A.* Fisher (p. 364). But the name and year, given in each of these places but one, are uniquely determinate.

There are a number of cases of names mentioned in Part I which are not represented in the Bibliography, at least in connection with items in question, for example: Wingquist (p. 22), Glaisher (p. 59), Wrench (p. 85), Stirling (p. 105), Atwood (p. 178), E. Wright (p. 185), Isaacson & Salzer (p. 202), Ikehara (p. 207), Norton, W. E. Deming and L. S. Deming (p. 209), Jeffreys (p. 212), Julia Bell (p. 219), B. A. Gould (p. 221), Blumer (p. 239), Sarmousakis (p. 242), Stoneley (p. 338), F. E. Allan (p. 364), Carse & Urquhart (p. 367), Buys-Ballot (p. 369), Darwin and Doodson (p. 370), Darwin-Börgen (p. 371).

S. A. JOFFE

B. Herewith are memoranda dealing with matters not of outstanding importance.

P. 83, 427. "Richter 1855," Richter died in 1854, and his 500D value of π first published in October 1854 (see *MTAC*, v. 2, p. 144), was reprinted in *Archiv* 1855. Hence "Richter 1854" is desirable here.

P. 112. The Duffield 1897 table is to 100 009, not 100 000.

P. 144. One wonders at the omission in **8.4** of a reference to Legendre's table of $\log\tan(45° + \frac{1}{2}x)$, *Traité des Fonctions Elliptiques*, v. 2, 1826, p. 256–259.

P. 377. According to the form of entry the authors declare that they have seen a copy of Bertrand's *Calcul des Probabilités*, dated 1888. The Brown University copy, dated 1889, and the *Catalogue* of the Bibliothèque Nationale give no hint of an 1888 edition.

P. 384. Where so many much less worthwhile items are listed one is surprised not to see a reference to OLIVER BYRNE, *Tables of Dual Logarithms*, London, Bell & Daldy, 1867.

P. 387. One might readily infer that the first English edition of Crelle's Calculating Tables was in 1908; the statement in FMR *Index* is "O. Seeliger's new edition first appeared . . . in English . . . in 1908," which is, of course, perfectly correct. The first English edition, however, was of Bremiker's revision of Crelle, London, Nutt, 1897. But much earlier there appeared *Tables for Facilitating the Operations of Multiplication, Division, and Evolution. Abridged from Dr. A. L. Crelle's Rechentafeln* by J. A. NORRIS, Washington, Govt. Printing Office, 1885. 20 p. Quarto format. [Text, p. 5–8; Tables p. 9–20.]

P. 390. The Desvallées, H. R. entry should be under Rocques-Desvallées, H.; see *Annuaire pour l'an* 1919 *publié par Le Bureau des Longitudes*, p. C4, and *Catalogue Général de la Librairie Française*, v. 29, p. 821.

P. 390. Why not refer to the 1934 reprint by Stechert, New York, of Dickson's *History of the Theory of Numbers*, v. 1, 1919?

P. 391. It would seem better to have expanded the title for Dupuis 1862 to become *Tables de Logarithmes à Sept Décimales d'après Callet, Véga, Bremiker, etc.*, so as to indicate their lack of originality.

P. 392. Surely it were especially desirable to give as a second reference for Encke 1832, his *Ges. math. u. astron. Abhandlungen*, Berlin, v. 3, 1889, p. 71–78.

P. 394. The second editions of GAUSS, *Werke*, v. 2, 1876, and v. 3, 1876, are not noted. D. H. LEHMER, *Guide to Tables in the Theory of Numbers*, p. 100, points out that v. 2, 1876, for example, is not merely a reprint, since it contains material not in v. 2, 1863.

P. 396. The authors inquire whether the tiny booklet by "Serge de Glasenapp," *Tables de Logarithmes . . .* , Paris, 1934, 127 p., 8.8 × 12.9 cm., is by Serge P. Glazenap, author of *Matematicheskie i Astronomicheskie Tablitsy*, Leningrad, 1932, iv, 241 p. 17.3 × 25 cm. The answer is yes.

P. 401. The second edition of Heger's *Fünfstellige logarithmische u. goniom. Tafeln*, is dated 1913, not 1914. In the title should be "*Hülfstafeln*" not "Hilfstafeln."

P. 407. The authors apparently state that they have seen Kennelly's *Chart Atlas*, third ed. revised and enlarged, dated "1926"; the Brown Univ. copy is dated 1924.

P. 408. Of Kepler's *Chilias Logarithmorum* 1624 there was also a so-called 1639 edition; but except for the first two leaves the editions are identical.

P. 410. Under L. "Leau" is "? *Tables des Parties Proportionelles*, Paris, Gauthier-Villars." This 10-page pamphlet by Léau (not Leau), see *Catalogue Général de la Librairie Française*, v. 29, p. 571, was published in 1921. Poggendorff lists this title and uses the form "Leau." "*Proportionnelles*" is mispelled by FMR.

P. 416. The entry under A. Meyer (a work not seen by the authors) is misleading. Anton Meyer, professor of mathematics at the Univ. of Liège, died in 1857. On the request of his widow his ms. *Théorie Analytique des Probabilités* was edited by his former student F. J. P. FOLIE, and published at Brussels in 1874. This was translated into German by EMANUEL CZUBER and published at Leipzig in 1879 under the title *Vorlesungen über Wahrscheinlichkeitsrechnung*.

P. 420. For Newcomb 1882, the authors list only the first edition of *Logarithmic and Other Mathematical Tables*. The dates of many other reprints up to 1921 are given in my Bibliography of Simon Newcomb, Nat. Acad. Sci., *Memoirs*, v. 17, 1924, p. 55. In 1921 this was the "best seller" of all of Newcomb's works on the market.

P. 420. The title of Newton's work is *Trigonometria Britanica* (not *Britannica*).

P. 427. The French edition, 1887 of F. W. Rex, *Fünfstellige Logarithmen-Tafeln* is listed. It might have been noted that the author's initials there become F. G.

P. 431. After the title of Schorr 1916 should be: *Publ. A-F.* Hamburg, Lucas Gräfe.

P. 431. Of Schrön 1860, there was an Italian edition in 1867 and a French edition in 1891.

P. 433. The entry Sherman 1933, should be under J. Sherman & L. Brockway; their little table is described elsewhere in this issue, N 62.

P. 434. The authors appear to state that they have seen a second edition of Spence's *Essay* dated 1820. As we have already noted, *MTAC*, v. 1, p. 458–459, the copy, which we consulted, of *Mathematical Essays* containing this, is dated 1819, not 1820.

P. 439. Since the authors have Chebyshev not Tschebyshev, for "Tschuprow" should there not be at least a cross-reference from Chuprov, even though the title-page has the form of name given?

P. 440. It seems somewhat remarkable that the authors do not list in the Bibliography the important publication of MAXIMILIAN VON LEBER, *Tabularum ad Faciliorem et Breviorem, in Georgii Vegae "Thesauri Logarithmorum" magnis Canonibus, Interpolationis Computationem utilium, Trias*. Vienna, 1897, 51 p., including the listing of thousands of Vega errors.

P. 443. Under Hoëne Wroński 1827 (see *MTAC*, v. 2, p. 18) it is stated that there was a Russian edition published at St. Petersburg in 1845. For this date the authors probably took as their excellent authority *Encycl. d. Sci. Math.*, I. 4. 2, 1908, p. 306. On the other hand, the very careful historian, S. DICKSTEIN, in his *Hoene Wroński*, Cracow, 1896, p. 328, gives the date of publication as 1844.

P. 444. The authors usually make a point of indicating the date of the first edition of each work listed, but have not done this in the case of L. ZIMMERMANN, *Vollständige Tafeln der Quadrate aller Zahlen*. This is the complete title of the work appearing at Liebenwerda, Verlag des technischen Versandsgeschäfts R. Reiss, 1898.

R. C. A.

C. By courtesy of Mr. D. H. SADLER, Superintendent of the Nautical Almanac Office, the authors have now seen J. W. CAMPBELL, *Numerical Tables of Hyperbolic and Other Functions*, Boston, etc., Houghton Mifflin, 1929. This useful little volume, which we vainly tried both to buy and to borrow during the war, is doubtless known to many of our American readers.

We find that the following is the only major correction required by the *Index:* Art. **10.41** (p. 170), *for* 0(.0001).025, and 0(.01)2.5 *read* 0(.0005).025, and 0(.05)2.5. A minor correction in Art. **7.12** (p. 123) is that Campbell's table of tan x gives 5 figures (instead of 4 decimals) for $x = 1.472(.001)1.670$, i.e. round $x = \frac{1}{2}\pi$. There are also trivial corrections of the nature that, for example, tables ending at $x = 2.999$ and 7.99 have been described as ending at 3 and 8 respectively. Campbell gives no differences, but gives proportional parts on a folding sheet at the end of the book.

If the major (and possibly the minor) correction is made, users of the *Index* may like to delete the asterisk in the J. W. Campbell item in Part II (p. 384).

ALAN FLETCHER

University of Liverpool

90. FRANCE, Service Géographique de l'Armée, *Tables des Logarithmes à huit Décimales*, Paris, 1891.

J. HENDERSON, *Bibliotheca Tabularum Mathematicarum*, 1926, states, p. 143, "It will be of interest to remark that this is the first 8-place table since 1658. Error: Log 28917 should be one unit less in the last place." FMR, INDEX, p. 112, states one unit more. In the table log 28917 = 4.46115 324. Since $28917 = 3^5 \cdot 7 \cdot 17$ it was easy to find, by means of Grimpen's table, that log 28917 = 4.46115 32349 908 is correct to 13D. Hence Henderson, and not FMR, is correct. This is the only known error in this table.

S. A. JOFFE

91. A. M. LEGENDRE, *Tafeln der Elliptischen Normalintegrale erster und zweiter Gattung*, hrsg. von FRITZ EMDE, Stuttgart, 1931. See MTE **86**, v. 2, p. 136 f.

In the previous survey we omitted to take account of three errors, noted by HEUMAN l.c., p. 144, occurring in the 1816 table of Legendre, but correct in Legendre's 1826 table. These are in $E(\phi, \theta)$ as follows:

Page	ϕ	θ	*For*	*Read*
385(50)	23°	51°	0.33502···	0.39502···
409(74)	14°	82°	0.241959···	0.341969···
413(78)	2°	86°	0.03489 9351	0.03489 9531

Heuman lists errors in log F (24°.9), log F (30°.9), log E (34°.4), log E (86°.0), in Legendre 1826, and therefore in Pearson. Only one of these errors occurs in Legendre 1816, namely in log E (34°.4), but this table was not reprinted by Emde. In Legendre 1826 there are also errors in F (73°, 5°), E (54°, 15°), F (35°, 30°), E (45°, 35°). The erratum, *MTAC*, v. 2, p. 137, l. 11, for F (35°, 30°) is to be deleted. Thus, there are at least 33 serious errors in the Emde and Potin Legendre tables, and at least 42 serious errors in the Pearson Legendre table. It may be noted that three of Legendre's errors previously listed were included by Heuman in a Supplementary Errata sheet published at Stockholm in 1941. Finally, on p. 137 (*l.c.*), *for* 1.17204, *read* 3.17204; and *for* 1.17024 *read* 3.17024; and in l. 4, *for* Seventeen, *read* Sixteen.

R. C. A.

92. NYMTP, "Table of $f_n(x) = n!J_n(x)/(x/2)^n$," *J. Math. Phys.*, v. 23, 1944. See *MTAC*, v. 1, p. 363.

P. 50, $n = 4$, $x = 8.5$, *for* .01527 8693, *read* .01527 8963. P. 52, $n = 6$, $x = 9.4$, $*\delta^2$, *for* 193518, *read* 193528. P. 59, $n = 19$, $x = 1$, *for* .98757 4123, *read* .98757 4124.

NYMTP.

93. U. S. Hydrographic Office, *Publication* no. 214, v. 4, 1940, *Tables of Computed Altitude and Azimuth, Latitudes* 30° to 39°, *inclusive*. Compare *MTAC*, v. 1, p. 81 f.

In order to decide whether the Japanese made use of H.O. No. 214 in the computation of the altitudes and azimuths given in their *Celestial Air Navigation Table*[1] (a problem which will not be considered in the present discussion), it was desirable to have a complete list of all errors in the altitudes in a specific section of the former tables. It was decided that the tabular material for the bright star Deneb as observed between 30°–39° North latitude would serve as a good basis for comparison. Hence the altitudes corresponding to declination 45°, declination same name as latitude, were computed for all integral degrees of local hour angles down to the horizon and for integral latitudes 30°–39°, with seven-place natural values of trigonometric functions. These altitudes were compared with those given (down to 5° altitude) in H.O. 214. The results of this comparison are given below:

L	Errors	Errors >0'.1	Tabular Values
30°	14	0	115
31°	58	16	117
32°	42	1	118
33°	57	2	120
34°	41	1	121
35°	60	12	123
36°	40	2	125
37°	36	2	126
38°	58	2	128
39°	45	4	130
Totals	451	42	1223

More than a third of the values are in error by at least one unit in the last place given and approximately 3.5 per cent are in error by two or more units. As will be seen below, the largest errors found were three of 0'.4 each. Though the sample examined is a small one, it is believed that it is fairly representative of the accuracy to be expected of H.O. 214.

The uneven distribution of errors among the various latitudes would seem to indicate that in the preparation of H.O. 214, different latitudes were assigned to different computers. For example, the work on latitude 30° is vastly superior to that on latitudes 31° and 35°.

Below is given a list of the 42 errata of two or more units in the last place:

$d = 45°$

	$L = 31°$			$L = 35°$	
t	h should be	error in 0'.1		h should be	error in 0'.1
5°	75°27'.8	2	11°	76°56'.9	2
9°	74°19'.8	2	12°	76°26'.8	3
10°	73°57'.8	2	13°	75°55'.3	2
14°	72°13'.8	2	14°	75°22'.5	2
15°	71°44'.4	4	17°	73°38'.2	2
16°	71°13'.8	2	18°	73°01'.8	3
17°	70°42'.1	2	25°	68°31'.1	2

t	h should be	error in 0'.1
19°	69°35'.9	2
21°	68°26'.4	2
22°	67°50'.6	2
24°	66°37'.2	2
25°	65°59'.7	2
29°	63°25'.1	2
31°	62°05'.7	2
36°	58°42'.5	2
39°	56°38'.3	2
	$L = 32°$	
15°	72°32'.5	2
	$L = 33°$	
12°	74°50'.3	2
17°	72°13'.2	2
	$L = 34°$	
15°	74°04'.9	2

t	h should be	error in 0'.1
27°	67°10'.3	2
29°	65°48'.5	2
31°	64°25'.9	2
35°	61°38'.9	2
52°	49°39'.4	2
	$L = 36°$	
11°	77°44'.1	2
12°	77°12'.6	2
	$L = 37°$	
11°	78°29'.4	2
12°	77°56'.4	2
	$L = 38°$	
20°	73°31'.5	2
27°	68°43'.2	2
	$L = 39°$	
5°	82°56'.8	3
6°	82°31'.8	4
7°	82°03'.9	4
28°	68°27'.9	2

I wish to acknowledge the valuable assistance of Miss EVELYN LINDSAY and Miss NANCY ARNOLD in the work of computing and checking the values used.

CHARLES H. SMILEY

Brown University

[1] See *MTAC*, v. 2, 1946, p. 44.

UNPUBLISHED MATHEMATICAL TABLES

The list of some tables prepared by The Radio Corporation of America, published below, suggests that many other mathematical tables must have been prepared during recent years at various research centers. The Editors would heartily welcome Reports on such tables. Other Unpublished Tables are referred to in RMT **320** (Strömgren). See also N61.

50[F].—ROBERT JAMES PORTER (1882–): *Tables giving the complete classification of primitive binary quadratic forms for negative determinants from* $-D = 2$ *to* $-D = 1000$. Ms. calculated during 1945 and the first quarter of 1946, the property of the author, residing at 266 Pickering Road, Hull, England.

The Ms. is in loose-leaf form, 298 pp. $8 \times 10\frac{1}{4}$ inches. The tables are in long-hand, in pencil, and consist principally of the main table, 259 pp., arranged in six parallel columns, the first containing the determinant number with its prime factors, in symbolic form; the second, the positive forms belonging to each determinant, arranged in ascending order of magnitude of the middle term; the third, the number of genera; the fourth, all the forms

arranged in periods; the fifth, their quadratic and supplementary characters, and the last, the composition, indicated by symbols.

There are several auxiliary tables which occupy 39 pp. These include (1) a list of primes to 1327, to facilitate the computation of the forms, (2) a table indicating by the signs plus and minus the quadratic residues and non-residues mutually appertaining to the first thirty or so primes, to simplify the work of determining the characters, (3) a table giving the composition of a large number of the simpler forms with each other, (4) a table giving the duplication of all the simpler forms in which the first term does not exceed 13, (5) a table showing the possible types of determinant factors with the number of genera each type affords, (6) a table from which the number of forms belonging to a determinant $-DS^2$ may be immediately computed from the number already obtained for the determinant $-D$, this being a valuable check on the working of the former, (7) a table of all determinants giving 1, 2, 4, 8, or 16 genera, according to the number of forms in each, and (8) a serial index of the results for each determinant.

Some observations on the results obtained may be of interest. The total number of forms listed in the main table is 15542. Of these, 1274 are given by 92 determinants of only one genus each, in which the longest period (of 45 forms) is produced by the determinant -971; 6100 forms are given by 402 determinants with 2 genera each (of which, for the determinant -941, each contains 23 forms); 6656 forms are produced by 417 determinants each with 4 genera, of which the most frequently appearing type is that where each genus contains 3 forms—this happens in 109 cases; 1496 forms are given by 87 determinants with 8 genera each, none containing more than 5 forms; and finally, there is only one determinant, namely, -840, which yields 16 genera, each of a single form.

The remaining point of interest is the discrepancy between the number of irregular determinants occurring in the present table, and the number given by Cayley in 1862. In addition to the thirteen there listed (i.e. $-D = 243$, 307, 339, 459, 576, 580, 675, 755, 820, 884, 891, 900, 974) it would appear that irregularity also exists where $-D = 468$, 544, 547, 931, 972. The author of this article will be glad to hear from any computer who can corroborate or correct this last statement.

R. J. Porter

Editorial Note: The most extensive tables of this kind previously published are in A. Cayley, "Tables des formes quadratiques binaires pour les déterminants négatifs depuis $D = -1$ jusqu'à $D = -100$, pour les déterminants positifs non carrés depuis $D = 2$ jusqu'à $D = 99$ et pour les treize déterminants négatifs irréguliers qui se trouvent dans le premier millier," *J. f. d. reine u. angew. Math.*, v. 60, 1862, p. 357–372; also in *Coll. Math. Papers*, v. 5, 1892, p. 141–156. See also A. E. Cooper, "Tables of quadratic forms," *Annals of Math.*, s. 2, v. 26, 1925, p. 309–316 for negative determinants for $-D = 101(1)200$.

51[L].—Radio Corporation of America, RCA Victor Division, *Tables of Integrals* in possession of the Corporation at Camden, New Jersey.

Our unpublished manuscripts of tables of integrals include the following:

1. $A(x) = (2/\pi)\int_0^x \tan^{-1} t\, dt/t$, $x = [0(.01).5; 8D]$.

2. $B(x) = (1/\pi)\int_0^x \ln\left|\frac{1+t}{1-t}\right|\frac{dt}{t} = (2/\pi)\int_0^x \tanh^{-1} t\, dt/t$, $x = [0(.01).97$-

$(.005).99(.002)1; 8D]$.

3. $C(u) = \frac{1}{2}\int_0^x J_{-\frac{1}{2}}(t)dt = \int_0^u \cos(\frac{1}{2}\pi t^2)dt$,

$S(u) = \frac{1}{2}\int_0^x J_{\frac{1}{2}}(t)dt = \int_0^u \sin(\frac{1}{2}\pi t^2)dt$,

$x = [0(.001).02(.01)2; 8D]$, $x = \frac{1}{2}\pi u^2$. These are the ordinary Fresnel integrals.

4. $Ch(u) = \frac{1}{2}\int_0^x I_{-\frac{1}{2}}(t)dt = \int_0^u \cosh\left(\frac{1}{2}\pi t^2\right)dt,$

$Sh(u) = \frac{1}{2}\int_0^x I_{\frac{1}{2}}(t)dt = \int_0^u \sinh\left(\frac{1}{2}\pi t^2\right)dt,$

$x = [0(.001).02(.01)2; 8D]$, $x = \frac{1}{2}\pi u^2$. These are the modified Fresnel integrals.

5. $K(x) = D_x^{\frac{1}{2}} \sin x = (1/\pi^{\frac{1}{2}}) \int_0^x \cos (x - t)dt/t^{\frac{1}{2}},$

$L(x) = D_x^{\frac{1}{2}} \sinh x = (1/\pi^{\frac{1}{2}}) \int_0^x \cosh (x - t)dt/t^{\frac{1}{2}},$

$M(x) = D_x^{\frac{1}{2}}(1 - \cos x) = (1/\pi^{\frac{1}{2}}) \int_0^x \sin (x - t)dt/t^{\frac{1}{2}},$

$N(x) = D_x^{\frac{1}{2}}(\cosh x - 1) = (1/\pi^{\frac{1}{2}}) \int_0^x \sinh (x - t)dt/t^{\frac{1}{2}},$

$x = [0(.02)1(.1)5; 8D]$. These are the one-half derivatives of Riemann and can be expressed in terms of Fresnel integrals and modified Fresnel integrals.

6. $H(x^{\frac{1}{2}}) = (1/\pi^{\frac{1}{2}}) \int_0^x e^{-t}dt/t^{\frac{1}{2}} = (2/\pi^{\frac{1}{2}}) \int_0^{\sqrt{x}} e^{-t^2}dt$

$= (2^{\frac{1}{2}}/\pi) \int_0^x K_{\frac{1}{2}}(t)dt = 2^{\frac{1}{2}}[Ch(u) - Sh(u)],$

$x = \frac{1}{2}\pi u^2 = [0(.001).02(.01)2; 8D]$. This is the error function for the argument $\sqrt{x}$.

7. $D(x^{\frac{1}{2}}) = (1/\pi^{\frac{1}{2}}) \int_0^x e^t dt/t^{\frac{1}{2}} = (2/\pi^{\frac{1}{2}}) \int_0^{\sqrt{x}} e^{t^2}dt$

$= 2^{\frac{1}{2}}[Ch(u) + Sh(u)].$

$x = \frac{1}{2}\pi u^2 = [0(.001).02(.01)2; 8D]$. This is closely related to Dawson's integral for the argument $\sqrt{x}$.

MURLAN S. CORRINGTON

Advanced Development Section

MECHANICAL AIDS TO COMPUTATION

23[Z].—D. R. HARTREE, "The ENIAC, an electronic calculating machine," *Nature*, v. 157, 20 April 1946, p. 527.

Concluding sentence: Its flexibility and speed of operation will make it possible to carry out many numerical calculations, in many fields of investigation, which without its assistance would have been regarded as much too long and laborious to undertake.

24[Z].—HARVARD UNIVERSITY, Computation Laboratory, *Annals*, v. 1: *A Manual of Operation for the Automatic Sequence Controlled Calculator.* Cambridge, Mass., Harvard University Press, 1946, xiv, 561 p. + 9 plates. 20 × 26.7 cm. $10.00.

This volume gives the first really scientific account of the Automatic Sequence Controlled Calculator, the first of the large all-purpose digital calculators developed during the war, and was prepared by a staff of 24, Comdr. H. H. AIKEN in charge, largely through the efforts of Lt. GRACE M. HOPPER. The volume consists of six chapters, a bibliography of numerical analysis, and a sizable appendix in eight parts. There are 16 full-page photographs showing parts of the machine.

Chapter 1, entitled Historical Introduction, gives a short account of early mechanical computers of the "difference engine" type, especially those associated with the names of

Pascal (1642), Leibnitz (1694), C. Babbage (1812–33), Scheutz (1834), Wiberg (1863), and Grant (1871).

Chapter 2 gives a description of the present calculator. In its outward appearance it consists of a 51 foot panel, 8 feet high, with two 6 foot panels extending at right angles from the back. The mechanical parts are driven in synchronism by a 4 H.P. motor. The machine weighs about five tons. The essential parts of the machine consist of mechanical counters and relays for the most part. They may be described very briefly as follows:

1. A register of 60 signed constants, of 23 digits each, manually set on switches. This provides the machine with data, such as coefficients of power series and small empirical tables, which are known in advance of the calculation being performed.

2. A set of 72 electro-mechanical adding-storage registers automatically operated by the machine. These 23-figure counters are for storing and combining those numbers which are produced by the machine in the course of its operation. The high capacity of this element of the machine is one of its best features.

3. A central multiplier and divider. These processes are performed by first building up a small table of the first nine multiples of the multiplicand or divisor and then dealing with this table appropriately.

4. Electro-mechanical tables of log x, 10^x and sin x. Logarithms are computed by the familiar method using four factors of the form $1 + h \cdot 10^{-k}$ ($k = 0(1)3$, $h = 0(1)9$) and the power series for log $(1 + x)$. A reversal of this process gives 10^x. Power series are used to compute either sin x or cos x when $x < \pi/4$, and all other values of the trigonometric functions may be easily derived from these.

5. Three tape-driven interpolators. Besides instructing the machine in the routine of some interpolation process (which may be as high as eleventh order) one of these units may be used to introduce numerical values into the machine.

6. A tape-driven sequence unit for controlling the whole machine. This is a 24-hole tape punched, in advance of the calculation, in three columns of eight holes each. Tapes as long as 5000 steps or rows are sometimes used. This unit is the salient feature of the machine and accounts for its name.

These are the internal parts of the calculator. The machine makes contact with the outside world through the following practically standard IBM equipment: two punch card readers, a card punch and two automatic typewriters. A manually operated tape punch for the preparation of tape for the four-tape units, four large plugboards and a number of control buttons make up the rest of the equipment.

The calculator normally handles 23 significant figures but can be set to deal with 12 or 46. Numbers are transferred from one part of the machine to another in the form of timed electrical pulses of 50 volts over a single buss. The simultaneous transfer of two or more numbers is thus impossible. However some parallel programming is possible during multiplication or division.

The fundamental unit of time in the machine is called the cycle. This is the time required to add two numbers or to move the control tape one step ahead and is three tenths of a second (200 per minute). Multiplication requires 20 cycles while the calculation of log x, 10^x, or sin x requires about a minute. No data are given on the time required to read or punch a card or to type one line. These operations, whose durations depend somewhat on the amount of data handled, are relatively slow since the cards are fed lengthwise.

The calculator is capable of a certain amount of discrimination. Register, or counter, No. 70, called the choice counter, can be used to change the sign of a quantity if and only if the number in the choice counter is negative. Counter No. 72 may be used to stop the machine when the number in it is less, in absolute value, than a preassigned tolerance. This is used in making automatic checks. Because the machine possesses but a single control tape, discrimination cannot be used to alter the routine of a complex calculation.

Chapter 3 gives an account of the electrical circuits of the calculator as they apply to the separate components. The discussion here is in simplified form. For a complete and accurate account of the circuits and relays as they really exist, the reader is referred to the appendix.

Chapter 4, entitled Coding, is the most difficult one to understand, chiefly because of the apparent lack of order or structure in the system. No doubt there is a code book (perhaps shown in plate XV) in which further explanations are disclosed. As it is, coding appears to be an almost unsurmountable barrier between the machine and the mere mathematician with a problem to solve. To begin with, one is dismayed to find that the 72 storage counters have code numbers as well as serial numbers. Thus counter number 48 has a code number of 65, while counter 29 has a code of 5431. However, this is the same early impatience that one finds in beginning to learn a foreign language. The reader is referred to the book for details of the design and construction of tapes for the many operations of the machine. It would appear that a great deal of care, acumen, and time is necessary to be sure that a tape is efficiently and correctly punched. Once an operational tape has been prepared, however, it may be used and stored for future use when the same type of problem comes up again. This feature makes possible the use of the machine to compute extensive tables of a given function "on the side" while tapes for other shorter programs of higher priority are being prepared. Tables computed in this way are being published as further volumes of the *Annals of the Computation Laboratory of Harvard University* (see RMT **335**).

Chapter 5 gives plugging instructions for the 12 units of the machine requiring plugboards. These latter apparently are of the standard IBM type (although no mention is made as to whether they are demountable or not) and are used to shift the decimal point and to delete digits. Plugboards are used also to control the typewriters and the card readers and punch.

Chapter 6, entitled Solution of Examples, gives 13 short examples seven of which are completely worked out to the point where they could be put on the machine. Example 7, for instance, consists in evaluating the two linear forms (quadrature formulae)

$$\Delta I_n = K(-f_{n-1} + 8f_n + 5f_{n+1}), \quad \overline{\Delta I}_n = K(5f_n + 8f_{n+1} - f_{n+2})$$

where $K = .000833333333$, given the values $f_0, f_1, \cdots, f_{500}$ on punch cards. The quantities ΔI and $\overline{\Delta I}$ are to be compared and $I = \sum \Delta I_n$ accumulated and typed. To get the problem started requires 93 cycles of coding on a beginning tape. Then a main control tape of 103 cycles is set on the machine and run for 500 revolutions. The whole calculation requires at most 205 minutes.

Obviously this is an important chapter for those readers who might want to adapt their problem to this calculator. One's zeal in attempting to master these techniques is apt to be dampened a little by the final paragraph of the chapter to the effect that the procedures outlined in the examples are already a year old and have been improved by permanent changes in the wiring of the calculator. There is no doubt that actual contact with the machine itself is the very best way to learn of its details.

Perhaps the most useful part of the volume, for the general reader, is its Bibliography of Numerical Analysis (p. 338–404). It "is composed principally of those references that have been found useful during the one and one-half years of operation of the Automatic Sequence Controlled Calculator." The material is arranged in 23 topics (four with subtopics) ranging from Historical Background to Integral Equations. This bibliography of about 1200 titles should prove immensely useful to any worker in the field of computation.

D. H. L.

25[Z].—E. LAURILA, "Ein Produktintegraph," Suomen Tiedeakatemia, Helsingfors, *Toimituksia*, *Annales*, s. A, *I. Mathematica-Physica*, no. 29, 1945, 12 p. 17.6 × 24.6 cm.

A linkage mechanism is described for evaluating the integral $\int_0^x f(x)\, g(x)\, dx$. While there is no particular novelty in the development, the simple design may be found attractive where a low-cost but relatively low-precision device is required.

S. H. C.

EDITORIAL NOTE: See V. BUSH, F. D. GAGE & H. R. STEWART, "A continuous integraph," Franklin Institute, *J.*, v. 203, 1927, p. 63–84. Among other things this instrument integrates the product of two functions, making use of the principle of electrical integrating watt hour-meter, combined with a moving table.

NOTES

59. ADMIRALTY COMPUTING SERVICE.—In *MTAC* we have recently had occasion more than once to refer to publications of this Service, for example, v. 2, p. 31, 35, 36, 39, 40, 80. "At the end of the War, the Admiralty Computing Service was providing a fairly comprehensive mathematical and computational service which was not only meeting all demands from Admiralty sources but was also able to offer informal assistance to the other Services, Government departments and contractors who had no comparable facilities at their disposal." This is a quotation from an article in *Nature*, v. 157, 4 May 1946, p. 571–573, entitled, "Mathematics in Government Service and industry. Some deductions from the war-time experience of the Admiralty Computing Service." Actual computations were carried out at the Nautical Almanac Office by a group under the direction of its Superintendent, D. H. SADLER, and "arrangements were made whereby scientific workers in the universities and elsewhere could be employed as consultants." Most of the members of the computing groups were recently transferred to the permanent National Mathematical Laboratory in the Mathematics Division of the National Physical Laboratory at Teddington. This Division is under the direction of Mr. J. R. WOMERSLEY. In view of our introductory article in the present issue of *MTAC* it is interesting to note in the *Nature* survey that "our experience fully confirms the statements made on many occasions by Dr. L. J. Comrie, . . . that full exploitation of the capabilities of the commercial calculating machines (including the National accounting machine and the Hollerith) is usually the most efficient way of dealing with problems, and specially designed calculating machines and instruments are necessary only for large-scale investigations of infrequent occurrence."

60. COEFFICIENTS IN AN ASYMPTOTIC EXPANSION FOR $\int_a^b e^{P(u)}du$.—The expansion for $\int_a^b e^{P(u)}du$ which is given below is well known in principle, since it is the result of merely continuing an integration by parts where at each successive step $e^{P(u)}du$ is replaced by $\dfrac{de^{P(u)}}{P'(u)}$. It has been employed extensively by several mathematicians of the NYMTP, especially for the purpose of extending the range of the usual asymptotic series by expressing the remainder $\int_a^b F(u)du$ as $\int_a^b e^{P(u)}du$. This method is often applicable to an integral of a real oscillatory function by considering it as either the real or imaginary part of a complex integral over the range a to b, provided that $P' \equiv P'(u)$ has no zeros in that range while $P^{(m)} \equiv P^{(m)}(u)$ are sufficiently small in comparison with P'.

The purpose of the expansion below is to enable one to obtain many terms of an asymptotic expansion, while avoiding the excessive labor of differentiating more and more complicated expressions that arise in repeat-

edly integrating by parts the various special functions that might occur in the integrand. It turns out to be much more expedient to work from this general expression in terms of the derivatives of the logarithm of the integrand. The following expansion was arranged as a series in $1/P'$, and it includes terms as far as the twelfth power[1] of $1/P'$:

$$\begin{aligned}\int_a^b e^{P(u)}du \sim {} & |P_1^{-1}e^P[1 + P_1^{-2}P_2 - P_1^{-3}P_3 + P_1^{-4}(P_4 + 3P_2^2) - P_1^{-5}(P_5 \\ & + 10P_3P_2) + P_1^{-6}(P_6 + 15P_4P_2 + 10P_3^2 + 15P_2^3) - P_1^{-7}(P_7 + 21P_5P_2 \\ & + 35P_4P_3 + 105P_3P_2^2) + P_1^{-8}(P_8 + 28P_6P_2 + 56P_5P_3 + 35P_4^2 \\ & + 210P_4P_2^2 + 280P_3^2P_2 + 105P_2^4) - P_1^{-9}(P_9 + 36P_7P_2 + 84P_6P_3 \\ & + 126P_5P_4 + 378P_5P_2^2 + 1260P_4P_3P_2 + 280P_3^3 + 1260P_3P_2^3) \\ & + P_1^{-10}(P_{10} + 45P_8P_2 + 120P_7P_3 + 210P_6P_4 + 630P_6P_2^2 + 126P_5^2 \\ & + 2520P_5P_3P_2 + 1575P_4^2P_2 + 2100P_4P_3^2 + 3150P_4P_2^3 + 6300P_3^2P_2^2 \\ & + 945P_2^5) - P_1^{-11}(P_{11} + 55P_9P_2 + 165P_8P_3 + 330P_7P_4 + 990P_7P_2^2 \\ & + 462P_6P_5 + 4620P_6P_3P_2 + 6930P_5P_4P_2 + 4620P_5P_3^2 + 6930P_5P_2^3 \\ & + 5775P_4^2P_3 + 34650P_4P_3P_2^2 + 15400P_3^3P_2 + 17325P_3P_2^4)]|_a^b,\end{aligned}$$

where P, $P'[= P_1]$, $P^{(2)}[= P_2]$, etc. are taken at the limits $u = b$ and $u = a$. It usually happens that $b = \infty$ and the expression vanishes at ∞.

To illustrate the effectiveness and convenience of this method, consider the ordinary asymptotic expansion for the error function in the form $F(z) \equiv e^{z^2}\int_z^\infty e^{-u^2}du$. It is well known that

$$F(z) = \frac{1}{2z} - \frac{1}{4z^3} + \frac{1\cdot 3}{8z^5} - \frac{1\cdot 3\cdot 5}{16z^7} + \cdots + \frac{(-1)^n 1\cdot 3\cdot 5 \cdots (2n-1)}{2^{n+1}z^{2n+1}} + R_{n+1},$$

where

$$R_{n+1} = (-1)^{n+1}\frac{1\cdot 3\cdot 5 \cdots (2n+1)}{2^{n+1}} e^{z^2} I_{n+1},$$

and where

$$I_{n+1} = \int_z^\infty \frac{e^{-u^2}du}{u^{2n+2}}.$$

Thus

$$I_{n+1} = \int_z^\infty e^{-u^2-(2n+2)\ln u}du, \quad P = -u^2 - (2n+2)\ln u,$$

$$P' = -2u - \frac{2n+2}{u}, \quad P^{(2)} = -2 + \frac{2n+2}{u^2},$$

and for $m > 2$,

$$P^{(m)} = \frac{(-1)^m(m-1)!(2n+2)}{u^m}.$$

Hence one obtains

$$\begin{aligned}I_{n+1} = \frac{e^{-z^2}}{2z^{2n+1}(z^2+n+1)}\Bigg[1 & + \frac{(-z^2+n+1)}{2(z^2+n+1)^2} - \frac{(n+1)}{2(z^2+n+1)^3} \\ & + \frac{3(z^4 - 2(n+1)z^2 + (n+1)(n+2))}{4(z^2+n+1)^4} \\ & + \frac{(n+1)(5z^2-5n-8)}{2(z^2+n+1)^5} + \cdots\Bigg].\end{aligned}$$

Suppose that $z = 3$ and $n = 9$. Then the ordinary asymptotic series will have a relative error of about $2 \cdot 10^{-4}$. But the first five terms of this expansion improve the accuracy by a factor whose relative error is about 10^{-5}, or in other words, about 5 additional places. For larger values of z and n, taking further terms for I_{n+1} would improve it by relatively much more. This expansion for I_{n+1} can be used for almost any real or complex value of z that is not too small, also provided that z is not "too near" either of the singular points $\pm i\sqrt{n+1}$.

HERBERT E. SALZER

NYMTP

[1] For convenience of printing, the order of the derivative has here been replaced by a subscript, so that the kth derivative raised to the lth degree, i.e. $P^{(k)l}$, is here denoted by P_k^l.

61. GUIDE (*MTAC*, no. 7), SUPPL. 4 (for Suppl. 1–3, see *MTAC*, v. 1, p. 403, v. 2, p. 59, 92).—The following entries are extracted from an unpublished ms. of 263 p., found in Darmstadt, Germany, *Verzeichnis berechneter Funktionen* by ERICH WILLI HERMANN KAMKE (1890–):

F. BORGNIS, "Die elektrische Grundschwingung des kreiszylindrischen Zweischichten-Hohlraums," *Hochfrequenztechnik u. Elektroakustik*, v. 59, 1942, p. 23.
Graphs for the least solutions y of

$$\frac{aJ_0(y)J_1(ay) - J_1(y)J_0(ay)}{aY_0(y)J_1(ay) - Y_1(y)J_0(ay)} = \frac{J_0(y/x)}{Y_0(y/x)},$$

for $x = .1, .25(.25)1; 0 \leqq a \leqq 9$.

K. FEDERHOFER, "Biegungschwingungen der in ihrer Mittelebene belasteten Kreisplatte," *Ingenieur-Archiv*, v. 6, 1935, p. 73. Solutions to 4D of the equations

$$ivJ_0(u)J_1(iv) - uJ_0(iv)J_1(u) = 0$$
$$v^2 - u^2 = m^2, \quad m = -4(1)0.$$

K. FEDERHOFER, "Berechnung der Auslenkung beim Ausbeulen dünner Kreisplatten," *Ingenieur-Archiv*, v. 11, 1940, p. 121–124.
Values to 4S of $U_1 = \int_0^{x_1} tJ_1^4(t)dt$, and $U_2 = \int_0^{x_1} J_1^4(t)dt/t$, $x_1 = 3.8317$, being the approximate first zero of $J_1(x)$.

G. FRANKE, "Die Theorie der Resonanzmembran," Siemens-Konzern, *Wiss. Veröffentl.*, v. 9, part 2, 1930, p. 162 and 164.

(a) The two smallest solutions of the equations

$$\tfrac{1}{4}\epsilon xk[J_0(xk)Y_0(x) - Y_0(xk)J_0(x)] = J_1(xk)Y_0(x) - Y_1(xk)J_0(x), \text{ and}$$

(b) The smallest solutions of the equations

$$J_0(xk)Y_1(x) - Y_0'(xk)J_1(x) = \left(\frac{1}{xk} - \frac{\epsilon}{4}xk\right)[J_1(xk)Y_1(x) - Y_1(xk)J_1(x)],$$

for $\epsilon = 0, 30, 50, 100$; $k = [0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}; 3D]$. Also figures for $\epsilon = 30, 50, 100, \frac{1}{6} \leqq k \leqq \frac{3}{4}$.

A. A. Gershun, "Berechnung des Volumleuchtens," *Physikalische Z. d. Sowjetunion*, v. 2, 1932, p. 180.

$$F(x) = 1 - \int_0^{\pi/2} e^{-x\cos t}\cos t\,dt = \tfrac{1}{2}\pi[\mathbf{H}_1(ix) - iJ_1(ix)]$$
$$= \tfrac{1}{2}\pi[I_1(x) - L_1(x)],$$

where

$$L_1(x) = -\mathbf{H}_1(ix) = \sum_{m=1}^{\infty} \frac{(x/2)^{2m}}{\Gamma(m+\frac{1}{2})\Gamma(m+\frac{3}{2})},$$

for $x = [0(.1)1(.2)2(.5)6(1)10;\ 4D]$.

K. Karas, "Eigenschwingungen inhomogener Saiten," Akad. d. Wissen., Vienna, *Sitzungsb.*, Abt. IIa, v. 145, 1936, p. 797–826, contains results as follows:

I. The first two solutions of the equations

$J_{1/3}(\frac{2}{3}\sqrt{x}) = 0$, $J_{1/4}(\frac{1}{2}\sqrt{x}) = 0$, $J_{\pm 1/3}(\frac{1}{6}\sqrt{2x}) = 0$, $J_{-2/3}(\frac{1}{6}\sqrt{2x}) = 0$, $J_{\pm 1/4}(\frac{1}{8}\sqrt{x}) = 0$, $J_{-3/4}(\frac{1}{8}\sqrt{x}) = 0$, to 3D or 5–6S, (21), (27), (61), (62), (64), (68), (69), (71), p. 803, 820.

II. The first two solutions, to 3D, of the equations

$$J_{1/3}(\tfrac{2}{3}\sqrt{x})J_{-1/3}(\tfrac{4}{3}\sqrt{2x}) - J_{-1/3}(\tfrac{2}{3}\sqrt{x})J_{1/3}(\tfrac{4}{3}\sqrt{2x}) = 0,$$
$$J_{1/3}(\tfrac{2}{3}\sqrt{x})J_{2/3}(\tfrac{1}{6}\sqrt{2x}) + J_{-1/3}(\tfrac{2}{3}\sqrt{x})J_{-2/3}(\tfrac{1}{6}\sqrt{2x}) = 0,$$
$$J_{1/3}(\tfrac{2}{3}\sqrt{x})J_{-1/3}(2\sqrt{3x}) - J_{-1/3}(\tfrac{2}{3}\sqrt{x})J_{1/3}(2\sqrt{3x}) = 0,$$
$$J_{1/3}(-\tfrac{2}{3}\sqrt{x})J_{2/3}(-\tfrac{1}{6}\sqrt{2x}) - J_{-1/3}(-\tfrac{2}{3}\sqrt{x})J_{-2/3}(-\tfrac{1}{6}\sqrt{2x}) = 0,$$
$$J_{1/3}(-\tfrac{2}{3}\sqrt{x})J_{-1/3}(-\tfrac{1}{6}\sqrt{2x}) + J_{-1/3}(-\tfrac{2}{3}\sqrt{x})J_{1/3}(-\tfrac{1}{6}\sqrt{2x}) = 0,$$

(23), (65), (66), (67), p. 803, 820.

III. The first two solutions, to 3D, of the equations

$$J_{1/4}(\tfrac{1}{8}\sqrt{x})J_{-1/4}(\tfrac{9}{8}\sqrt{x}) - J_{-1/4}(\tfrac{1}{8}\sqrt{x})J_{1/4}(\tfrac{9}{8}\sqrt{x}) = 0,$$
$$J_{1/4}(\tfrac{1}{2}\sqrt{x})J_{-1/4}(2\sqrt{x}) - J_{-1/4}(\tfrac{1}{2}\sqrt{x})J_{1/4}(2\sqrt{x}) = 0,$$
$$J_{1/4}(\tfrac{1}{2}\sqrt{x})J_{-1/4}(\tfrac{9}{8}\sqrt{x}) - J_{-1/4}(\tfrac{1}{2}\sqrt{x})J_{1/4}(\tfrac{9}{8}\sqrt{x}) = 0,$$
$$J_{1/4}(\tfrac{9}{8}\sqrt{x})J_{3/4}(\tfrac{1}{2}\sqrt{x}) + J_{-1/4}(\tfrac{9}{8}\sqrt{x})J_{-3/4}(\tfrac{1}{2}\sqrt{x}) = 0,$$

(29), (72), (73), (74), p. 803, 820. Of the third equation only the first solution is given.

Iwao Kobayashi, "Das elektrostatische Potential um zwei auf derselben Ebene liegende und sich nicht schneidende gleichgrosse Kreisscheiben," Sendai, Tôhoku Teikoku Daigaku, *Science Reports*, s. I, v. 27, 1939, p. 387–391. There are tables of

$$g(\lambda, \mu, \nu, p) = \frac{1}{\sin\frac{1}{2}\pi(\mu+\nu-\lambda)} \int_0^{\infty} J_\lambda(pt)J_\mu(t)J_\nu(t)\,dt/t,$$

$p = [2(.2)3(.5)4, 5, 7, 10; 7D]$
$\lambda = 0(1)5$; $\mu = \frac{1}{2}(1)2\frac{1}{2}$; $\nu = \frac{1}{2}(1)5\frac{1}{2}$, $\mu \leqq \nu$, $\mu + \nu \leqq 6$; λ and $\mu + \nu$ not simultaneously even or uneven; otherwise all combinations of λ, μ, ν. Also

$$H_{mh}^{kn} = (2m + 4n + 1)g(h + m, m + 2n + \tfrac{1}{2}, h + 2k + \tfrac{1}{2}; p),$$

$$H_{oh}^{kn} = (4n + 1)g(h, 2n + \tfrac{1}{2}, h + 2k + \tfrac{1}{2}; p),$$

for $p = [2(.2)3(.5)4, 5, 7, 10; 5D]$, $k = 0(1)2$, $n = 0(1)2$, $m = 0(1)5$, $h = 0(1)5$.

B. VAN DER POL & H. BREMMER, "The propagation of radio waves over a finitely conducting spherical earth," *Phil. Mag.*, s. 7, v. 25, 1938, p. 823.
$H_\nu^{(1)}[\frac{1}{3}(-2x)^{\frac{3}{2}}] = 0$, $\nu = \frac{1}{3}$ and $\frac{2}{3}$, the first 6 zeros to 4S.

S. TOMOTIKA, "The instability of a cylindrical column of a perfect liquid surrounded by another perfect fluid," Nippon Suugaku-buturigakkwai Kizi, Tokyo, *Proc.*, s. 3, v. 18, 1936, p. 559–561.
$F(x) = (1 - x^2)x/[\{K_0(x)/K_1(x)\} + AI_0(x)/I_1(x)]$, to 5–6D, for $A = 5$, 1, .2, .1; $x = 0(.1)1$. Also $A = 5$ and 1, $x = .6(.02).8$; and $A = .2$ and .1, $x = .5(.02).7$.

P. F. WARD, "The transverse vibrations of a rod of varying cross-section," *Phil. Mag.*, s. 6, v. 25, 1913, p. 89, 94, 97, 100, 101, 103. We here find zeros, to 5S, of the following functions:

$$\frac{d}{dx}[J_0(2\sqrt{x})I_0(2\sqrt{x})], \text{ first 4;}$$

$$\frac{d}{dx}\left[\frac{1}{x}J_1(2\sqrt{x})I_1(2\sqrt{x})\right], \text{ first 3;}$$

$$\frac{d}{dx}\left[\frac{d}{dx}\left\{\frac{1}{\sqrt{x}}J_1(2\sqrt{x})\right\}\frac{d}{dx}\left\{\frac{1}{\sqrt{x}}I_1(2\sqrt{x})\right\}\right], \text{ first 4;}$$

$$\frac{d}{dx}\left[\frac{d}{dx}\left\{\frac{1}{x}J_2(2\sqrt{x})\right\}\frac{d}{dx}\left\{\frac{1}{x}I_2(2\sqrt{x})\right\}\right], \text{ first 3;}$$

$$I_1(2\sqrt{x})\frac{d^2}{dx^2}\left[\frac{1}{\sqrt{x}}J_1(2\sqrt{x})\right] - J_1(2\sqrt{x})\frac{d^2}{dx^2}\left[\frac{1}{\sqrt{x}}I_1(2\sqrt{x})\right], \text{ first 3;}$$

$$I_2(2\sqrt{x})\frac{d^2}{dx^2}\left[\frac{1}{x}J_2(2\sqrt{x})\right] - J_2(2\sqrt{x})\frac{d^2}{dx^2}\left[\frac{1}{x}I_2(2\sqrt{x})\right], \text{ first 3.}$$

K. YOSIKATA, "Values of the functions $\bar{K}_0$ and $\bar{K}_1$," Sendai, Tôhoku Teikoku Daigaku, *Technology Reports*, v. 9, 1929, p. 347 f.

$kI_0(x) - K_0(x)$ and $kI_1(x) + K_1(x)$,
$k = \ln 2 - C = .11593\cdots$, $x = [.1(.1)5; 10D]$,
$x = [5(.1)6(1)11; 8\text{–}9S]$, $x = [.02(.02)1; 7D]$.

62. TABLES OF $(\sin x)/x$, AND OF SOME OF ITS FUNCTIONS.—Such tables are desirable for certain applications in diffraction phenomena and in a

variety of other fields as the following references suggest. Augmentation of the list would be welcomed.

1. K. Hayashi, *Tafeln der Besselschen, Theta-, Kugel- und anderer Funktionen*, Berlin, 1930, p. 30–47.
[0(.01)10(.1)20(1)100; 8D].
2. *Comisión Impulsora y Coordinadora de la Investigación Científica, Anuario 1944*. Mexico, 1945, p. 223–247. See *MTAC*, v. 2, p. 121.
[0(.001)1; 7–8S].
3. N. R. Jørgensen, *Undersøgelser over Frekvensflader og Korrelation*, Diss., Copenhagen, 1916, p. 159–165. The form of statement of the tabulation was suggested by FMR *Index*.
[$0(2^g)2000^g$; 7D], Δ.
4. BAASMTC, *Mathematical Tables*, v. 2, *Emden Functions* . . . , London, 1932, p. 1, $y = (\sin x)/x$. Suggested by FMR *Index*.
[0(.1)3.f; 7D]. $y^{(n)}$, $n = 1(1)5$ are also tabulated for the same x.
5. E. C. J. v. Lommel, "Die Beugungserscheinungen geradlinig begrenzter Schirme," Bayer. Akad. d. Wissen., *math. natw. Abt., Abh.*, v. 15, 1886, p. 651.
[0(1)50; 6D]. $(\sin^2 x)/x^2$ is also tabulated for the same ranges. There are also 17 6D values of maxima and minima of $(\sin x)/x$ and $(\sin^2 x)/x^2$.
6. S. H. Bauer, "The function of sin x/x," Optical So. Amer., *J.*, v. 22, 1932, p. 537.
[0(.1)1.6, $\frac{1}{2}\pi$, .5(.5)50; 5D]. This table was "checked by a large scale plot, all points having been found to fall on a perfectly smooth curve."
7. A. Schuster, *Introduction to the Theory of Optics*, London, 1904, p. 101; third ed., 1924, p. 105.
[0(15°)540°; 4D]. $(\sin^2 x)/x^2$ is also tabulated for [0(15°)180°; 4D], [180°(15°)540°; 5D].
8. J. Sherman & L. Brockway, "A four place table of $(\sin x)/x$," *Z. f. Kristallographie*, v. 85A, 1933, p. 404–419.
[0(.01)20(.02)40(.05)100; 4D].
For $x < .25^r$, $y = (\sin x)/x$ was calculated by means of a Taylor's expansion about the origin. For $.25 < x < 6$, Peters' six-place table of natural sines was employed, and for x between 6 and 100 radians Lohse's five-place table of natural sines was basic. The table was checked by computing first and second differences throughout. When the second difference indicated probability of an error in y greater than a unit in the last place, the value of y was recomputed.
9. G. Bruhat, *Cours d'Optique*. Paris, Masson, 1930, p. 216; also second ed. 1935, p. 216.
[$0(\pi/12)4\pi$; 4D].
10. E. Jahnke & F. Emde, *Tables of Functions with Formulae and Curves*, fourth ed., New York, 1945, Addenda, p. 32–35. There's a graph of the function, p. 35, $0 < x < \pi$; also a table of max. and min. values on p. 30. See *MTAC*, v. 1, p. 203, 235, **E**, 6, 7, 8.
[0(.01)3.14; 4–5S].
11. R. Heger, *Fünfstellige logarithmische und goniometrische Tafeln sowie Hülfstafeln zur Auflösung höherer numerischer Gleichungen*, second ed. Leipzig and Berlin, 1913. T. 26, p. 85.
[20°(5°)50°(2°)90°; 4D].

12. K. STUMPFF, *Tafeln und Aufgaben zur harmonischen Analyse und Periodogrammrechnung*. Ann Arbor, 1944, p. 122–126; see *MTAC*, v. 2, p. 32. Also $x \csc x$ and $x^2 \csc^2 x$, $x \leqslant 100^g$; and $x \csc x$, $101 \leqslant x \leqslant 200$. $[1^g(1^g)200^g(2^g)2000^g(10^g)8000^g$; 4D]. Also an auxiliary table (p. 126) for calculating the function, $8000^g \leqslant x \leqslant 12000^g$.
13. °J. ENBERG & E. LÅNGSTRÖM, *Tables à six Décimales des Valeurs Naturelles des Fonctions Trigonométriques*. Stockholm, 1925. $(\sin x)/x^g$ and $x^g \csc x$. $[0(0^g.01)15^g$; 6S].
14. E. WEBER, "Die magnetischen Felder in leerlaufenden Synchronmaschinen," *Archiv f. Elektrotechnik*, v. 19, 1927, p. 197. There are graphs of $[\sin \frac{1}{2}\nu\pi x]/\frac{1}{2}\nu\pi x$, $\nu = 1(2)19$, $0 \leqslant x \leqslant .55$.
15. We have already surveyed various tables for $S = \log [(\sin x)/x]$, *MTAC*, v. 1, p. 83–85. See also v. 2, p. 125, RMT **306** (G).
16. FRIEDRICH RISTENPART & WILHELM EBERT, tables in P. HARZER, *Ueber die Bestimmung und Verbesserung der Bahnen von Himmelskörpern nach drei Beobachtungen*. Kiel, Sternwarte, *Publ.*, v. 11, 1901, p. 109–116. There is here a table of $\log [x \csc x]$ for $\log x = [\bar{2}.36(.001) + .04$; 7D], Δ.
17. S. GRADSTEIN, "Erzwungene Torsionsschwingungen von Kurbelwellen," *Ingenieur-Archiv*, v. 3, 1932, p. 212–214; $x \csc x$, $x = [0(.05)10$; 5D].
18. J. T. PETERS, *Sechsstellige Tafel der trigonometrischen Funktionen . . . von zehn zu zehn Bogensekunden* . . . Berlin, 1929 and 1939, p. 24–31; Russian eds. 1937 and 1938; $x'' \csc x$ $[0(10'')1°20'$; 6S]; also, p. 22, critical values up to 1°20′10″.
19. J. T. PETERS, *Sechsstellige trigonometrische Tafel für neue Teilung*, Berlin, 1930, and 1939, p. 44. $x \csc x^g$, to 4D, up to 2^g.
20. L. J. COMRIE, *Four-Figure Tables . . . with Argument in Time*. London, 1931, p. 32, $x \csc x$ to $1055^s.5$.
21. W. O. LOHSE, *Tafeln für numerisches Rechnen mit Maschinen*, second ed. by P. V. Neugebauer. Leipzig, 1935, p. 15. $x \csc x°$, $[0(0°.1)3°$; 6S].
22. J. T. PETERS, *Sechsstellige Werte der Kreis- und Evolventen-Funktionen* . . . Berlin and Bonn, 1937, p. 182. $x° \csc x°$ up to 1°.011. Reprinted in W. F. VOGEL, *Involutometry and Trigonometry*, Detroit, 1945, p. 182.
23. C. K. SMOLEY, *Segmental Functions. Text and Tables*. Scranton, Pa., 1943, p. 250–255, $\frac{1}{2}x \csc \frac{1}{2}x$, $x = [10°(1')30°(10')180°$; 7D].
24. Kempe's *The Engineer's Year-Book . . . for 1929*, v. 36. London, 1929, p. 58, 60, 62, $y = \frac{1}{2}x \csc \frac{1}{2}x$, and $\log y$, for $x = 1°(1°)180°$; 6 and 7D]. These tables are not in the later volumes.
25. In *Nature*, v. 120, 1927, p. 478, 770, 918, there are discussions by V. NAYLOR and A. E. LEWIN, of methods of solution of the equation $(\sin x)/x = c$. From tables indicated above, it is clear that we can readily read off the approximate real solutions. The case $c = 1$ has been discussed by A. P. HILLMAN & H. E. SALZER in *Phil. Mag.*, s. 7, v. 34, 1943, p. 575, where the first ten solutions with positive real and imaginary parts are given to 6D. For the same authors' discussion of the case $c = -1$ and also for a table by J. FADLE, $c = \pm 1$, see *MTAC*, v. 2, p. 60–61.

26. Of the *sine integral*, $\int_0^x \sin t\, dt/t$, there are many tables, including NYMTP, *Tables of Sine, Cosine and Exponential Integrals*, 2 v., 1940; *Table of Sine and Cosine Integrals for Arguments from* 10 *to* 100, 1942, $x = [0(.001)10(.01)100;\ 10D]$, $[0(.0001)1.9999;\ 9D]$.

GIBBS CONSTANT. The Fourier series $\sum_{n=1}^{\infty} \frac{\sin nt}{n}$ converges for each t, and has the values $\frac{1}{2}(\pi - t)$ for $0 < t < \pi$, $\frac{1}{2}(t - \pi)$ for $-\pi < t < 0$, 0 for $t = 0$ or π. Thus convergence is non-uniform in any interval including the point $t = 0$. The approximating curves $y_n = \sum_{\nu=1}^{n} \frac{\sin \nu t}{\nu} = f_n(t)$ behave in a strange way, namely near zero the nth curve stays above the highest point $\frac{1}{2}\pi$ of the straight line by a definite ratio, $K = \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t}\, dt$, observed by the physicist Gibbs and known as Gibbs constant; see *Nature*, v. 59, 1898, p. 200 and 1899, p. 606, or *Collected Works*, v. 2, 1928, part 2, p. 258–260. If the maximal value of $f_n(t_n)$ is taken at the point t_n so that

$$\max_{t<\pi} \sum_{\nu=1}^{n} \frac{\sin \nu t}{\nu} = \sum_{\nu=1}^{n} \frac{\sin \nu t_n}{\nu},$$

then for $t_n \downarrow 0$,

$$\lim_{n\to\infty} \sum_{\nu=1}^{n} \frac{\sin \nu t_n}{\nu} = \int_0^{\pi} \frac{\sin t}{t}\, dt = k,$$

which is the highest limiting point of the approximating curves and lies on the y-axis. This value is given as 1.85, approximately, in C. A. STEWART, *Advanced Calculus*, London, 1940, p. 431. Gibbs noted only that K existed but did not derive a numerical value. On the suggestion of OTTO SZÁSZ, in the Spring of 1944, the Harvard Automatic Sequence Controlled Calculator was used to find $K = 1.17897975\cdots$,[1] and this value is recorded in his paper "The generalized jump of a function and Gibbs' Phenomenon," *Duke Math. J.*, v. 11, 1944, p. 824. In two earlier papers he had given the incorrect value $1.08949\cdots$; Amer. Math. So., *Trans.*, v. 53, 1943, p. 440, and v. 54, 1943, p. 497, this incorrect value being taken from A. ZYGMUND, *Trigonometrical Series*, Warsaw-Lemberg, 1935, p. 180, as $1.089490\cdots$. The approximate value of k is given by S. A. COREY, *Amer. Math. Mo.*, v. 13, 1906, p. 13, as 1.851936—, which checks with Stewart's value noted above, but exhibits the notable inaccuracy of the value given in G. H. HARDY & W. W. ROGOSINSKI, *Fourier Series*, Cambridge, 1944, p. 36, $1.71\cdots$. This inaccurate value is that, $1.089\cdots \times \frac{1}{2}\pi$, given by Z. ZALCWASSER, *Fundamenta Mathem.*, v. 12, 1928, p. 127. The value of K given in P. FRANKLIN, *A Treatise on Advanced Calculus*, New York, 1940, p. 511, being $1.17\cdots$, is not far wrong. Anyone inclined to state that Gibbs was the first to observe his phenomenon should consult a little paper by H. WILBRAHAM, "On a certain periodic function," *Camb. and Dublin Math. J.*, v. 3, 1848, p. 198–201. For assistance in collecting material for this note **26** I am indebted to OTTO SZÁSZ, professor of mathematics at the Univ. of Cincinnati, and to PHILIP FRANKLIN, professor of mathematics at the Mass. Inst. of Technology. The first general and scientific discussion of the phenomenon was by M. BÔCHER, *Annals Math.*, s. 2, v. 7, 1906, p. 123f. See also H. S.

CARSLAW, (a) *Introd. to the Theory of Fourier's Series and Integrals*, 3rd ed. rev. and enl., London, 1930, p. 293–296; (b) "A historical note on Gibbs' phenomenon in Fourier's series and integrals," Amer. Math. So., *Bull.*, v. 31, 1925, p. 420–424; and also *Encycl. d. math. Wissen.*, v. II.3.2, p. 1203f.

R.C.A.

[1] D. H. L. writes as follows: This value has a last-figure error; in fact $K = 1.1789\ 79744\ 47216\ 72702\ 32029$. It is interesting to note that Zygmund's value is $\frac{1}{2}(K + 1)$. A value of $k = Si(\pi)$ is given to 16*S* in NYMTP, *Table of Sine, Cosine and Exponential Integrals*, v. 2, 1940, p. 206. From this it may be seen that Corey's value, referred to later, is also in error in the last figure, *for* 6 −, *read* 7. $k = 1.851\ 937051\ 982466$.

QUERY

19. THE INTEGRAL $\int_0^x e^{-A \sec \theta} d\theta$.—This integral arises in radium therapy discussion, and since tables of the function are so important for calculating the intensity of rod-shaped preparations, ROLF M. SIEVERT published such tables in his memoir, "Die *v*-Strahlungsintensität an der Oberfläche und in der nächsten Umgebung von Radiumnadeln," *Acta Radiologica*, Copenhagen, v. 11, 1930, p. 249–301. The tables on p. 271–280 are for $x = 30°(1°)90°$, $A = [.1(.01).5; 3D]$. In the recent work, C. W. WILSON, *Radium Therapy, its Physical Aspects*, London, Chapman & Hall, 1945, p. 213–214, there is an abridgment of these tables for $x = 30°(1°)90°$, $A = [0(.05).4; 3D]$. Current work connected with integrated radiation from a line source of radioactive material suggests the great desirability of extension of Sievert's table for $x < 30°$, and for $A > .5$. Have other tables of the integral been published?

ROBLEY D. EVANS

Dept. of Physics
Massachusetts Institute of Technology

QUERIES—REPLIES

25. BRIGGS' ARITHMETICA LOGARITHMICA (Q7, v. 1, p. 170; QR21, v. 2, p. 94).—In the library of the University of Michigan is a copy of this volume with the extra 12 pages described in the query.

LOUIS C. KARPINSKI

Univ. of Michigan

26. SCARCE MATHEMATICAL TABLES (Q2, v. 1, p. 66; QR5, p. 100; 6, p. 132).—Four libraries have already been noted where HENRY GOODWYN, *A Table of the Circles*. . ., 1823, may be consulted. We may now add that copies are also available in the libraries of Brown University and of L. J. C.

CORRIGENDA

V. 1

P. 215, B_6 1, *for* $s = 1(1)50$, *read* $s = 1(1)150$.
P. 220, A_1 1, *for* 15D, *read* 15–20D; *for* 21.5, *read* 25.5. A_2 1, *delete* δ^2.
P. 221, A_3 5, *for* $x/8$, *read* $1/(2x)$.
P. 223, B_2 5, *for* .5(.1)1, *read* .5, .6, .8(.1)1.
P. 226, A_1 4, *delete* Δ; A_1 8, *for* 0(.01)1, *read* 0(.01)5.1.
P. 229, B_3 10, *for* $(9 + x^3)$, *read* $(9 + x^3)^{\frac{1}{2}}$.

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CLASSIFICATION OF TABLES, AND SUBCOMMITTEES

A. Arithmetical Tables. Mathematical Constants
B. Powers
C. Logarithms
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E. Hyperbolic and Exponential Functions
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CONTENTS

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